Runge Kutta Methods

Trapezoidal rule
- $O(h^2)$
- implicit
- A-stable (see HW)

Midpoint rule
- $O(h^2)$
- explicit
- 2-step (needs bootstrap)
- negligible stability region: $z \in [-i, i]$.

Runge-Kutta methods are different from LMMs.

- All single step
- To obtain higher order, intermediate stages within the step are employed.

1-stage: only Euler's method

2-stage

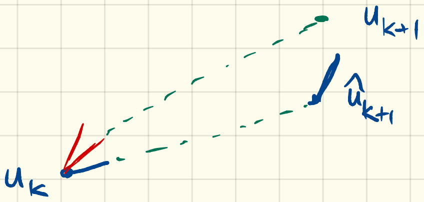
Consider the trapezoidal rule:

$$u_{k+1} = u_k + \frac{h}{2}\left[f(t_k, u_k) + f(t_k + h, u_{k+1}) \right]$$

Instead:

$$\hat{u}_{k+1} = u_k + h\, f(t_k, u_k)$$

$$u_{k+1} = u_k + \frac{h}{2}\left[f(t_k, u_k) + f(t_k + h, \hat{u}_{k+1}) \right]$$



This is an explicit method. "pseudo trapezoidal"

$O(h^2)$

Proof that method is $O(h^2)$

$$\frac{u_{k+1}-u_k}{h} - \frac{1}{2}\left[f(t_k,u_k) + f(t_k+h, u_k + h f(t_k,u_k))\right]$$

Now substitute true solution

$$\frac{u(t_k+h)-u(t_k)}{h} - \frac{1}{2}\left[f(t_k,u(t_k)) + f(t_k+h, u_k + h f(t_k,u_k))\right]$$

Now expand as a function of h

$$\frac{u(t_k+h)-u(t_k)}{h} = u'(t_k) + \frac{u''(t_k)h}{2} + O(h^2)$$

$$= f(t_k,u_k) + \frac{u''(t_k)h}{2} + O(h^2)$$

$$u''(t_k) = f_t + f_u u' = f_t + f_u f \quad (\text{all at } t_k, u_k)$$

$$= f + (f_t + f_u f)\frac{h}{2} + O(h^2)$$

$$g(h) = f(t_k + h,\ u_k + h f(t_k, u_k))$$

$$g(0) = f(t_k, u_k)$$

$$g'(h) = f_t(t_k + h,\ u_k + h f(t_k, u_k)) + f_u(t_k + h, u_k + hf) f$$

$$g'(0) = f_t(t_k, u_k) + f_u(t_k, u_k) f(t_k, u_k)$$

$$= f_t + f_u f$$

$$g(h) = f + (f_t + f_u f) h + O(h^2)$$

$$-\tau = f + (f_t + f_u f)\frac{h}{2} + O(h^2) - \frac{1}{2}\left[f + f + (f_t + f_u f) h + O(h^2)\right]$$

$$= O(h^2) \checkmark$$

Most general 2-stage R-K

$$Y_1 = U_k + h\left[a_{11} f(t_k + c_1 h, Y_1) + a_{12} f(t_k + c_2 h, Y_2)\right]$$

$$Y_2 = U_k + h\left[a_{21} f(t_k + c_1 h, Y_1) + a_{22} f(t_k + c_2 h, Y_2)\right]$$

$$U_{k+1} = U_k + h\left[b_1 f(t_k + c_1 h, Y_1) + b_2 f(t_k + c_2 h, Y_2)\right]$$

Y_1 is an estimate for u at $t + c_1 h$

Y_2 is an estimate for u at $t + c_2 h$

$$\left.\begin{aligned} a_{11} + a_{12} &= c_1 \\ a_{21} + a_{22} &= c_2 \\ b_1 + b_2 &= 1 \end{aligned}\right]$$

~~arise from consistency~~

Randy says this, but I don't believe it

$$\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$$

For this to be explicit

$$Y_1 = U_k$$

$$Y_2 = U_k + h\left[a_{21} f(t_k + c_1 h, Y_1) \right]$$

$$U_{k+1} = U_k + h\left[b_1 f(t_k + c_1 h, Y_1) + b_2 f(t_k + c_2 h, Y_2) \right]$$

Five free parameters $a_{21}, c_1, c_2, b_1, b_2$

$$c_1 = 0$$

$$a_{21} = c_2$$

$$b_1 + b_2 = 1$$

two free parameters.

Use this freedom to try to maximize the order of the method.

$$g(h) = b_1 f(t_k + c_1 h, u_k) + b_2 f(t_k + c_2 h, u_k + h a f(t_k + c_1 h, u_k))$$

$$g(0) = b_1 f(t_k, u_k) + b_2 f(t_k, u_k)$$

$$g'(0) = b_1 f_t c_1 + b_2 \left[f_t c_2 + f_u [a f] \right]$$

$$b_1 + b_2 = 1$$

$$g(h) = f + [f_t + f_u f] \frac{h}{2} + O(h^2)$$

desire

$$b_1 c_1 + b_2 c_2 = \frac{1}{2}$$

$$b_2 a = \frac{1}{2}$$

$$c_1 + b_2 (c_2 - c_1) = \frac{1}{2}$$

$$b_2 a = \frac{1}{2}$$

Pick $b_2 \neq 0$. $a = \frac{1}{2 b_2}$

Pick c_1 $\quad b_1 = 1 - b_2$

$$c_2 = \frac{1}{b_2} \left[\frac{1}{2} - b_1 c_1 \right]$$

$$b_2 \left[f_t c_2 + f_u a f \right]$$

$c_1 = 0$: $\quad b_1 + b_2 = 1$

$$b_2 c_2 = \frac{1}{2}$$

$$b_2 a = \frac{1}{2} \Rightarrow a = c_2$$

$$b_1 f(t_k, u_k) + b_2 f(t_k + ah, u_k + ah f(t_k, u_k))$$

$$b_1 + b_2 = 1$$

$$a b_2 = \frac{1}{2}$$

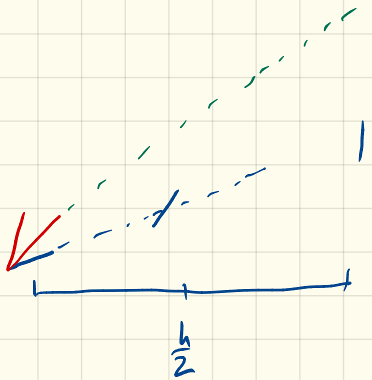
Variations

1) $a = 1$, $b_1 = b_2 = 1/2$

↳ pseudo trapezoidal from the start

2) $a = \frac{1}{2}$, $b_2 = 1$, $b_1 = 0$

↳ improved Euler



This looks like a real gem

- $O(h^2)$
- explicit
- single step (multi stage)

The drawback over trapezoidal: absolute stability.

$$u_{n+1} = u_n + h\left[b_1 \lambda u_n + b_2 \lambda (u_n + ah\lambda u_n)\right]$$

$$= u_n \left[1 + b_1 z + b_2 z + b_2 a z^2\right] \qquad z = h\lambda$$

$$= u_n \left[1 + z + \frac{1}{2} z^2\right]$$ (regardless of the method)

$$1 + z + \frac{1}{2} z^2 = 1$$

$$z + \frac{1}{2} z^2 = 0$$

$$z\left(1 + \frac{z}{2}\right) = 0 \qquad z = 0, \; z = -2,$$ same as Euler.

See workbook on website for a computation of stability region.

$$p(z) = 1 + z + \frac{1}{2}z^2$$

$$\underbrace{\left| 1 + z + \frac{1}{2}z^2 \right|}_{f(x+iy)} \leq 1$$

$f(x+iy)$ and look at contour of 1

<u>The</u> Runge-Kutta method is $O(h^4)$ with 4 stages.
RK4

You can think of it as inspired by Simpson's Rule

$$u_{n+1} = u_n + \frac{h}{6}\left[f(t_n) + 4 f\left(t_n + \frac{h}{2}\right) + f(t_n + h) \right] + O(h^5)$$

$$4 f\left(t_n + \frac{h}{2}\right) \to 2 f\left(t_n + \frac{h}{2}\right) + 2 f\left(t_n + \frac{h}{2}\right)$$

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

$$Y_1 = U_k$$

$$Y_2 = U_k + \frac{1}{2} f\left(t_k + \frac{h}{2}, Y_1\right)$$

$$Y_3 = U_k + \frac{1}{2} f\left(t_k + \frac{h}{2}, Y_2\right)$$

$$Y_4 = U_k + f\left(t_k + h, Y_3\right)$$

$$U_{k+1} = U_k + \frac{h}{6}\left[f(t_k, Y_1) + 2f(\overset{+\frac{h}{2}}{t_k}, Y_2) + 2f(\overset{+\frac{h}{2}}{t_k}, Y_3) + f(\overset{+h}{t_k}, Y_4)\right]$$

Exercise: plot the stability region of RK4.