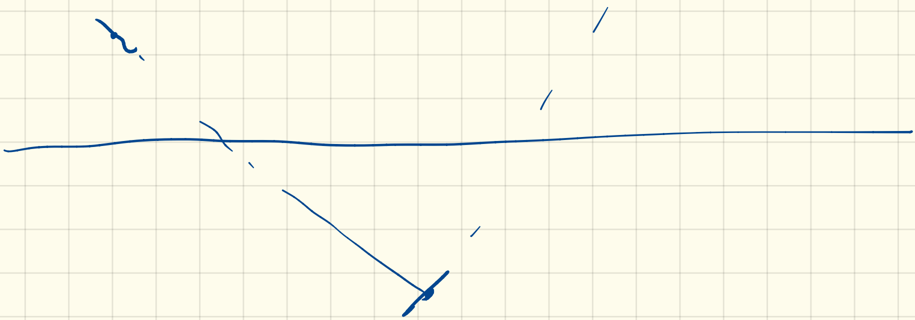
Last class:

For Eulers method applied to $u' = \lambda u$

We have convergence as $h \to 0$ (the method is zero stable).

But for large step sizes, we see a form of instability:



This looks like a form of instability.

Of course, you can eventually beat it by taking h small enough. We know Euler's method is convergent, so there can't be a fundamental theoretical problem. If you take h small enough, you will win.

But it can manifest itself as a practical problem. A key scenario involves "transients." Part of the solution is evolving on a large time scale, and part is decaying on a very short time scale.

e.g. 1)

$$y' = -20\left(y - \sin(t)\right) + \cos(t)$$

$$y(t) = \sin(t) + Ae^{-20t}$$

$$y' = \cos(t) - 20Ae^{-20t}$$

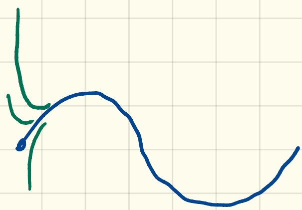
$$-20\left(y - \sin(t)\right) + \cos(t) = -20\left(Ae^{-20t}\right) + \cos(t)$$

$$= \cos(t) - 20Ae^{-20t}$$

$\sin(t)$ ↑ changes on scales $O(1)$

Ae^{-20t} ↑ decaying on scales $O(\frac{1}{20})$

Show worksheet.



e.g.

Consider $$u' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -40 \end{bmatrix} u$$

$$u = \begin{bmatrix} Ae^{-t} \\ Be^{-10t} \\ Ce^{-20t} \end{bmatrix}$$

(For linear systems, this is less special than it looks).

More generally, $u' = Au$ $\quad A$ $n \times n$.

If x is an eigenvector of A with eigenvalue λ,
$u = e^{\lambda t} x$ solves $u' = Au$.

If you have two negative eigenvalues, their different sizes give you different time scales of decay.

See worksheet. The different modes are evolving each on their own scale.

e.g. Robertson's reaction model

$$x' = -0.04x + 10^4 yz$$
$$y' = 0.04x - 10^4 yz - 3\times10^7 y^2$$
$$z' = 3\times10^7 y^2$$

$B \longrightarrow C$ quickly $(\sim 10^{-7})$

$B + C \longrightarrow A$ medium $(\sim 10^{-4})$

$A \longrightarrow B$ slowly $(\sim 10^{-2})$

The underlying issue is known as "stiffness", which requires two processes, one evolving on a slow scale and one decaying on a fast scale.

To apply Euler's method you are restricted to a time step dictated by the fast scale.

But those modes are decaying, and if you do a great job resolving them, that might be wasted effort. You need a method that resolves the slow scales but at least lets the fast scale stuff you are not resolving decay to 0.

Euler's method applied to $u' = \lambda u$

$$u_n = (1+\lambda h)^n u_0$$

If $\lambda < 0$ but $|1+\lambda h| > 1$, we'll see the solution oscillate and grow.

$$(1+\lambda h)^2 = 1$$

$$(\lambda h)^2 + 2\lambda h = 0$$

$z = \lambda h \qquad z = -2, 0$

z in here and at least the solution decays.

For reasons related to eigenvalues of systems, we'll consider $\lambda \in \mathbb{C}$, $\lambda < 0$.

Absolute stability

Apply the LMM to

$$u' = \lambda u \quad \text{with } Re(\lambda) < 0 \ (\text{text: } u' = -ru).$$

The method is <u>absolutely stable</u> for a step size h if for any initial data, $u_i \longrightarrow 0$.

Why consider $\lambda \in \mathbb{C}$?
Think of λ as an eigenvalue of a system.

Euler's method:

$$u_{n+1} = (1 + h\lambda) u_n$$

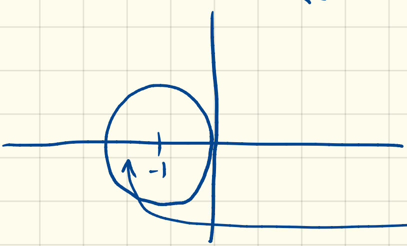
$$= (1+h\lambda)^n u_0$$

So solution $\to 0$ $\iff$ $|1 + h\lambda| < 1$

$\iff$ $|h\lambda - (-1)| < 1$

$\iff$ $h\lambda$ is in the circle of radius 1 of -1.

$z = h\lambda$



region of absolute stability.

The idea is that if h lies in the region of absolute stability, we will at least see decay even if we fail to well resolve the transient.

Backwards Euler

$$u_{n+1} = u_n + \underbrace{\lambda h}_{z\text{, always will occur for LMMs.}} u_{n+1}$$

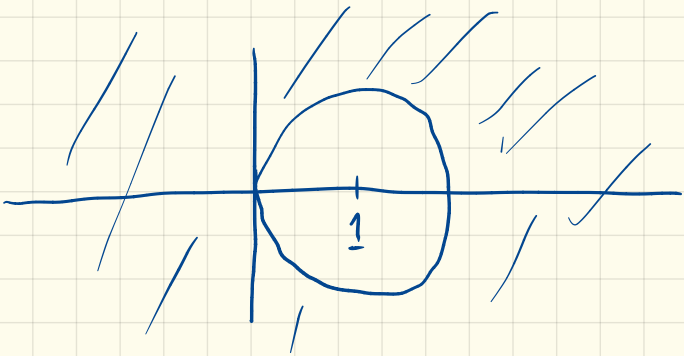
Restriction will always be on $z = h\lambda$

$$(1-\lambda h) u_{n+1} = u_n$$

$$u_{n+1} = \frac{1}{1-\lambda h} u_n$$

$$= (1-z)^{-1} u_n$$

We want $\left|(1-z)^{-1}\right| < 1 \iff |1-z| > 1$



In particular, we have decay if $\lambda < 0$ with any step size

(But restrictions on growth!)

Show worksheet.

Key point:

We can take time steps good enough to resolve the slow component of the solution.

We will do a relatively crappy job of resolving the transient

<u>But</u> the transient will decay to zero.
Eventually, it won't matter anyway (so why try resolving it?)

If your enemy is transients, you need to worry about absolute stability.

If not, you don't. You need to know about your system.

For a LMM

$$\alpha_k u_{n+k} + \cdots + \alpha_0 u_n = h\left(\beta_k f_{n+k} + \cdots + \beta_0 f_n\right)$$

$$\alpha_k u_{n+k} + \cdots + \alpha_0 u_n = \lambda h\left[\beta_k u_{n+k} + \cdots + \beta_0 u_n\right]$$

$$(\alpha_k - z\beta_k) u_{n+k} + \cdots + (\alpha_0 - z\beta_0) u_n = 0$$

Stability polynomial:

$$p(\rho) = (\alpha_k - z\beta_k)\rho^k + \cdots + (\alpha_0 + z\beta_0)$$

$$= \sigma(\rho) - z\left(\beta_k \rho^k + \cdots + \beta_0\right)$$

characteristic polynomial, from zero stability.

An LMM is <u>absolutely stable</u> for $z = \lambda h$ if every root of $p(\rho)$ satisfies $|\rho| \le 1$, and every root with $|\rho| = 1$ is simple. — root condition

Backwards Euler

$$\rho - 1 = \overbrace{\lambda h}^{z} \rho$$

$$(1 - z)\rho = 1$$

$$\rho = \frac{1}{1 - z}$$

Forwards Euler

$$\rho - 1 = \lambda h$$

$$\rho = 1 + z$$

abs stable if $|1 + z| < 1$.

Units of λ $[\lambda] = [t]^{-1}$ so $[t\lambda] = 1$

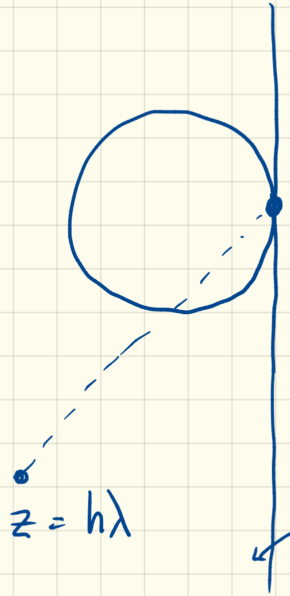
which makes λ meaningful to compare against real numbers.

Def: The absolute stability region of a LMM is the set of all z such that

the roots of $\rho_z(\rho)$ satisfy the

root condition.

For fixed λ, what happens as $h \to 0$?



suggests Euler's method will have trouble with nearly imaginary eigenvalues.

As you decrease h, z moves along this line and eventually enters the absolute stability region.

$$\rho^2 - 1 = 2z\rho$$

$$\rho^2 - 2z\rho - 1 = 0$$

$$\rho = \frac{2z \pm \sqrt{4z^2+4}}{2}$$

$$= z \pm \sqrt{z^2+1}$$

Product of roots is 1. Best case: both have norm 1.

E.g. Trapezoidal rule

$$u_{n+1} = u_n + \frac{h}{2}\left[f_n + f_{n+1} \right]$$

$$u_{n+1} = u_n + \frac{h}{2}\left[\lambda u_n + \lambda u_{n+1} \right]$$

$$\left(1 - \frac{z}{2}\right) u_{n+1} - \left(1 + \frac{z}{2}\right) u_n = 0$$

$$\left(1 - \frac{z}{2}\right) \rho - \left(1 + \frac{z}{2}\right) = 1$$

$$\rho = \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}}$$

$|\rho| = 1$ when $\left|1 - \frac{z}{2}\right|^2 = \left|1 + \frac{z}{2}\right|^2$

$$\left(1 - \frac{z}{2}\right)\left(1 - \frac{\bar{z}}{2}\right) = \left(1 + \frac{z}{2}\right)\left(1 + \frac{\bar{z}}{2}\right)$$

$$1 - \frac{z + \bar{z}}{2} + \frac{|z|^2}{4} = 1 + \frac{z + \bar{z}}{2} + \frac{|z|^2}{4}$$

$$z + \bar{z} = 0$$

$z = -\bar{z}$ → imaginary axis

<1 >1

Def: A method is A-stable

the method is absolutely stable for any choice

of z, $Re(z) < 0$.

Backwards Euler

Trapezoidal

Bad news (Dahlquist Barrier Theorem)

All A-stable LMMs are implicit, and have order of convergence of at most $O(h^2)$. [Of these, the trapezoidal rule has the smallest error coeff].

In fact, that was the 2nd barrier theorem

First: A zero stable q-step method has order of convergence of at most

$q+1$ if q is odd

$q+2$ if q is even

1 → 2

2 → 4

3 → 4

4 → 6

5 → 6

trapezoidal rule!