So Euler's method is zero stable with $K = 1$.

E.g., Mid point method

$$u_{i+1} - u_{i-1} = 0$$

Again, seek a solution ρ^i

$$\rho^{i+1} - \rho^{i-1} = 0$$

$$\rho^{i-1}(\rho^2 - 1) = 0$$

$\rho = \pm 1$ will work

$$u_0 = A + B \qquad \left(A = \frac{u_0 + u_1}{2},\ B = \frac{u_0 - u_1}{2} \right)$$

$$u_1 = A - B$$

$$u_n = A + (-1)^n B$$

So Midpoint method is zero stable with $K = 2$.

Last class: zero stability

Apply your method (a LMM) to $u' = 0$.

For a k-step method you'll need k initial conditions $u_0, \dots, u_{k-1}$.

Zero stable if $\exists K$, independent of h,

$$|u_n| \leq K \max(|u_0|, \dots, |u_{k-1}|)$$

Found $K = 1$ works for forward Euler, $K = 2$ for midpoint (leapfrog).

LMM reduces to

$$\alpha_k u_{n+k} + \dots + \alpha_0 u_k = 0$$

Def: The characteristic polynomial of an LMM

is $\alpha_k \rho^k + \cdots + \alpha_1 \rho + \alpha_0$.

$\rho - 1$ for Euler's method

$\rho^2 - 1$ for midpoint method

Thm: An LMM is zero stable iff

1) The characteristic polynomial has no roots ρ with $|\rho| > 1$

<u>and</u>

2) Any root of the char poly with $|\rho| = 1$ is a simple root.

What can go wrong if $|\rho| > 1$?

Is a solution $u_n = \varepsilon \rho^n$ $\qquad$ $u_M = \varepsilon \rho^M$, unbounded in M.

approximating $u' = 0$.

On your HW: is an example of a consistent method that is <u>not</u> zero stable.

Why the legalese?

$$u'' - 2u' + 1 = 0 \qquad e^{\lambda t}$$

$$(\lambda^2 - 2\lambda + 1)e^{\lambda t} = 0$$

$$(\lambda - 1)^2 e^{\lambda t} = 0 \rightarrow \lambda = 1.$$

Not enough: te^t is another:

$$u' = e^t + te^t$$

$$u'' = 2e^t + te^t$$

$$u'' - 2u' + u' =$$

$$te^t + 2e^t - 2e^t - 2te^t + te^t$$

$$= 0$$

General solution: $Ae^t + Bte^t$

In the same way

$$x_{n+2} - 2x_{n+1} + x_n = 0 \qquad \rho^n$$

$$\rho^n \left[\rho^2 - 2\rho + 1 \right] = 0 \qquad \rho = 1 \text{ only.}$$

$x_k = k\rho^k$ is another $x_k = k$ for us:

$$k + 2 - 2(k+1) + k = 0 \checkmark$$

If $|\rho| = 1$, $|k\rho^k| \rightarrow \infty$.

If $|\rho| < 1$, $|k\rho^k| \rightarrow 0$

$x_k = A\rho^k + Bk\rho^k = A + Bk$ is general solution.

As a consequence, we can't expect convergence.

Errors introduced by truncation can grow by a factor that is increasingly large as $h \to 0$, $M \to \infty$.

Compare $\left|(1+\lambda h)^i \tau_i\right| \le e^{|\lambda| T} |\tau_i|$

independent of M.

Theorem: (Dahlquist)

A consistent k-step LMM is convergent if and only if it is zero stable.

Moreover, if the truncation error is $O(h^p)$ and if $\left|u_i - u(t_0 + ih)\right|$ is $O(h^p)$ $0 \le i \le k-1$

Then the error is $O(h^p)$.

Good news: consistent 1-step methods are always 0-stable

$$\alpha_1 u_{i+1} - \alpha_0 u_i = h(\beta_1 f_{i+1} + \beta_0 f_i)$$

$$-\tau = \frac{\alpha_1 u(t_i+h) - \alpha_0 u(t_i)}{h} - \left(\beta_1 u'(t_i+h) + \beta_0 u'(t_i)\right)$$

$$= \frac{\alpha_1\left(u(t_i) + u'(t_i)h + O(h^2)\right) - \alpha_0 u(t_i)}{h} - \left(\beta_1 u'(t_i) + O(h) + \beta_0 u'(t_i)\right)$$

$$= \left[\frac{\alpha_1 - \alpha_0}{h}\right] u(t_i) + \left[\alpha_1 - \beta_1 - \beta_0\right] u'(t_i) + O(h)$$

So we need 1) $\alpha_1 - \alpha_0 = 0$

2) $\alpha_1 = \beta_1 + \beta_0$

$\alpha_1 \rho - \alpha_1 = \alpha_1(\rho - 1)$ only root is $\rho = 1$.

So is zero stable

Exercise: An LMM is consistent iff

1) $\alpha_k + \cdots + \alpha_0 = 0$

2) $k\alpha_k + \cdots 1\alpha_1 + 0\alpha_0 = \beta_k + \cdots + \beta_0$.

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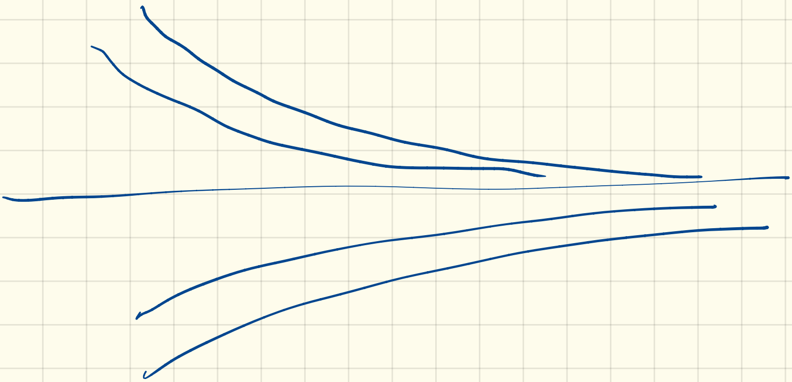
2) (Our second notion of stability)

Euler's method applied to

$$u' = \lambda u \qquad \lambda < 0.$$

Solutions $u(t) = Ce^{\lambda t}$, decaying exponentials



$$u_{i+1} = u_i + h\lambda u_i$$

$$= (1+h\lambda) u_i$$

Suppose $h > \frac{2}{|\lambda|}$

$$\lambda h < 0$$

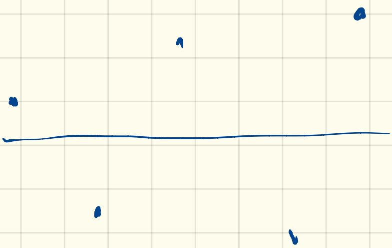
$$-\lambda h = |\lambda| h > 2$$

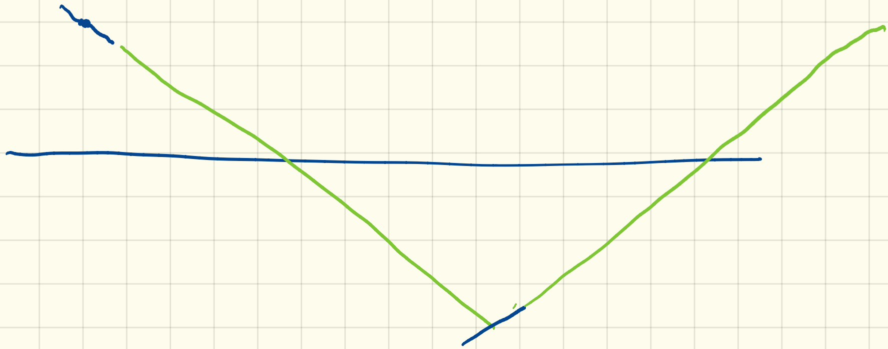
$$\lambda h < -2$$

$$1+\lambda h < -1$$

Solution: $u_0 (1+\lambda h)^j$

↳ sign changes and growing





If step size is too big, we overshoot, and if way big and can oscillate, or oscillate and grow.

This looks like a form of instability.

Of course, you can eventually beat it by taking h small enough. We know Euler's method is convergent, so there can't be a fundamental theoretical problem. If you take h small enough, you will win.

But it can manifest itself as a practical problem. A key scenario involves "transients." Part of the solution is evolving on a large time scale, and part is decaying on a very short time scale.