I

0.	0.
0.0	0.00
0.00	0.0000
0.000	0.000000
0.0000	0.00000000

in some number of time steps

Minor subtlety:

$$u(t_{i+1}) = u(t_{i-1}) + 2h\, f(t_i, u_i)$$

"multistep method". Information from two prior steps is used. It's still explicit, which is nice.

Need to bootstrap u_0, given and u_1, some other.

Need to pick u_1 to not spoil $O(h^2)$, and Euler's method will work: $h \cdot \tau$ error is $O(h^2)$.

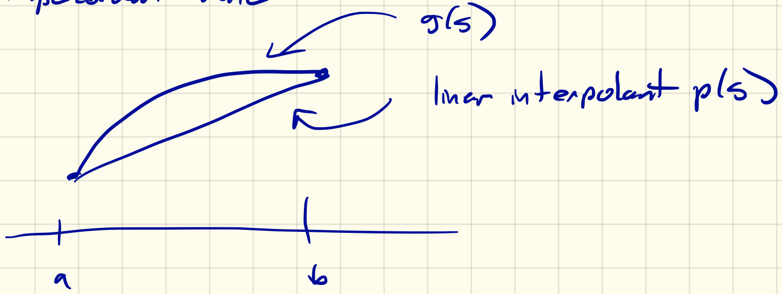
We'll shortly see some undesired behavior, though.

Methods obtained from quadrature

$$u' = f(t, u(t))$$

$$u(t_{i+1}) = u(t_i) + \int_{t_i}^{t_{i+1}} f(t, u(t))\,dt$$

4) Trapezoidal rule:



$$g(s) = p(s) + e(s) \qquad |e| \leq \frac{(b-a)^2}{8} \max |g''|$$

$$\int_a^b g(s) = \int_a^b p(s)\,ds + O((b-a)^3)$$

$$p(t) = f(t_i, u(t_i))\,\frac{(t_{i+1} - t)}{h} + f(t_{i+1}, u(t_{i+1}))\,\frac{t - t_i}{h}$$

$$\int_{t_i}^{t_{i+1}} p(t)\,dt = \frac{h}{2}\left[f(t_i, u(t_i)) + f(t_{i+1}, u(t_{i+1})) \right]$$

$$u_{i+1} = u_i + h\left[\frac{f(t_i, u_i) + f(t_{i+1}, u_{i+1})}{2}\right]$$

$$u(t_i+h) = u(t_i) + u'(t_i)h + \frac{u''(t_i)}{2} h^2 + O(h^3)$$

$$\frac{u'(t_i)+u'(t_i+h)}{2} = \frac{u'(t_i) + u'(t_i) + u''(t_i)h + O(h^2)}{2}$$

$$= u'(t_i) + \frac{u''(t_i)h}{2} + O(h^2)$$

Right hand side:

$$u(t_i) + u'(t_i)h + \frac{u''(t_i)h}{2} + O(h^3)$$

$$\frac{u_{i+1} - u_i}{h} - \left[\frac{f_i + f_{i+1}}{2}\right] = \underbrace{O(h^2)}_{LTE}$$

Method is implicit, single step, $O(h^2)$

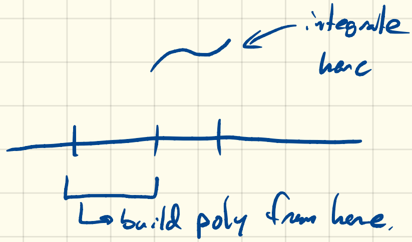
5)

Adams methods:

$$u_{i+1} = u_i + \int_{t_i}^{t_{i+1}} \underbrace{f(t, u(t))}_{\downarrow} \, dt$$

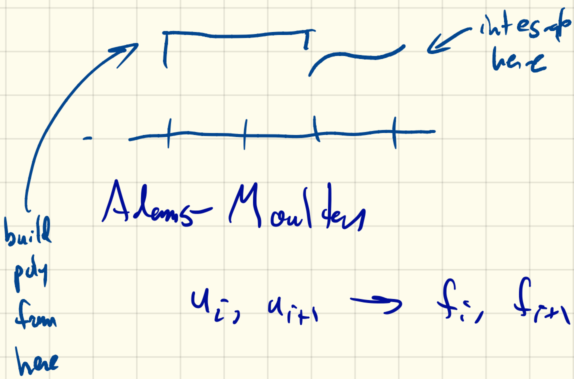
replace with a polynomial

Adams - Bashforth



$u_{i-1}, u_i \rightarrow \underbrace{f_{i-1}, f_i}$ → Build a linear polynomial and integrate it.

$u_{i-2}, u_{i-1}, u_i \rightarrow f_{i-2}, f_{i-1}, f_i$



Build a quadratic poly and integrate it.

Adams-Moulton

$u_i, u_{i+1} \rightarrow f_i, f_{i+1}$ Build a linear poly and integrate it

(Trapezoidal Rule)

$u_{i-1}\quad u_i\quad u_{i+1} \to f_{i-1},\ f_i,\ f_{i+1}$ Build a quadratic and integrate it.

Adams-B are explicit

Adams-M are implicit (inc

all are multistep (need bootstrapping)

Expect higher order convergence as number of steps goes up.

All methods so far

$$\alpha_k u_{i+k} + \cdots + \alpha_0 u_i = h\left(\beta_k f_{i+k} + \cdots + \beta_0 f_i\right)$$

$$f_j = f(t_j, u_j).$$

for certain constants $\alpha_0, \dots, \alpha_k$, $\beta_0, \dots, \beta_k$

These are known as Linear Multistep Methods

explicit: $\beta_k = 0$ single step $k=1$

Notions of stability

1) Right hand side $= 0$

Def: An LMM is zero stable if there is a constant K such that for a right-hand side of 0 and initial data $u_0, \ldots, u_{k-1}$

independent of M

$$|u_i| \le K \max_{0 \le j \le k-1} |u_j|$$

e.g. Euler's method:

$$u_{n+1} - u_n = 0$$

(We could just solve, of course...)

Linear recurrence relation, seek solution $u_i = \rho^i$

$$\rho^{n+1} - \rho^n = 0$$

$$\rho^n [\rho - 1] = 0$$

So $\rho = 1$. Every solution $C\rho$ $C \in \mathbb{R}$, So Euler's method is zero stable with $K = 1$.

E.g., Mid point method

$$u_{i+1} - u_{i-1} = 0$$

Again, seek a solution ρ^i

$$\rho^{i+1} - \rho^{i-1} = 0$$

$$\rho^{i-1}(\rho^2 - 1) = 0$$

$\rho = \pm 1$ will work

$$u_0 = A + B \qquad \left(A = \frac{u_0 + u_1}{2},\ B = \frac{u_0 - u_1}{2} \right)$$

$$u_1 = A - B$$

$$u_n = A + (-1)^n B$$

So Midpoint method is zero stable with $K = 2$.