Ordinary Differential Equations

In one variable, a first order ode is

$$u' = f(t, u)$$

Typically supplemented with an initial condition

$$u(t_0) = u_0 .$$

IVP, together.

We will frequently take $t_0 = 0$. (WLOG)

e.g. $u'(t) = g(t)$

$u'(t) = \lambda u + g(t)$

$u'(t) = \lambda(t) u + g(t)$

linear

$u'(t) = \lambda u(1-u)$ logistic

Thm: Suppose $f(t,u)$ is Lipshitz in u in a neighborhood of (t_0, u_0). I.e, there is a constant Λ such that

$$|f(t,u_1) - f(t,u_2)| < \Lambda |u_1 - u_2| \quad \text{for all } t, u_1, u_2$$

close to t_0, u_0. Then there is an $\varepsilon > 0$ and a unique function u on $(t_0 - \varepsilon, t_0 + \varepsilon)$ such that $u' = f(t,u)$.

Why ε?

$u' = u^2$

$u(0) = 1$

$u(t) = \frac{1}{1-t} \qquad u' = \frac{-1}{(1-t)^2}\cdot(-1) = \frac{1}{(1-t)^2} = u^2$ ✓

Only good up to $t = 1$

Blows up in finite time.

Why Lipschitz? Uniqueness can fail.

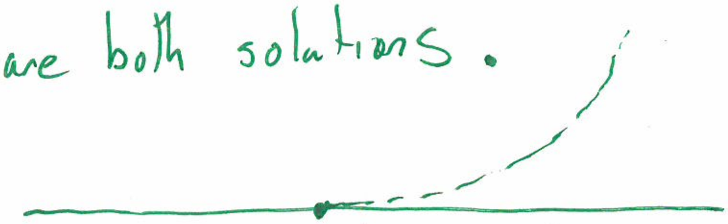
Exercise: Consider $u' = u^{1/3}$

$u(0) = 0.$

Show $u_1 \equiv 0$ and

$$u_2 = \begin{cases} 0 & t \le 0 \\ \left(\frac{2}{3}t\right)^{3/2} & \end{cases}$$

are both solutions.



Then find 8 others. Now convince yourself Lip. is violated.

cu^p

$pcu^{p-1} = u^{p/3} c^{\frac{1}{3}}$

$p = c^{-\frac{2}{3}}$

$c = p^{-\frac{3}{2}}$

$p - 1 = p/3$

$3p - 3 = p$

$2p = 3$

$p = 3/2$

$\left(\frac{2}{3}\right)^{3/2}$

$c = 1/4$

Exercise: Show $u = 0$ and $u = \begin{cases} \frac{1}{4}t^2 & t > 0 \\ 0 & t \le 0 \end{cases}$

are two solutions of $u' = \sqrt{|u|}$

with $u(0) = 0$.

Explain why this does not violate the previous theorem.

at^2

$2at$

$\sqrt{a}\,|t|$

$2a\,t$

$2a = \sqrt{a}$

$2\sqrt{a} = 1$

$4a = 1$

$a = \frac{1}{4}$

More generally, we have first order systems

$$\vec{u}\,' = \vec{f}(t, \vec{u})$$; I'll drop the arrows.

One might think of higher order equations as an omission, but these can be turned into systems:

$$x'' + cx' + kx = 0$$

$$v = x'$$

$$v' + cv + kx = 0$$

$$x' = v \qquad\qquad \begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

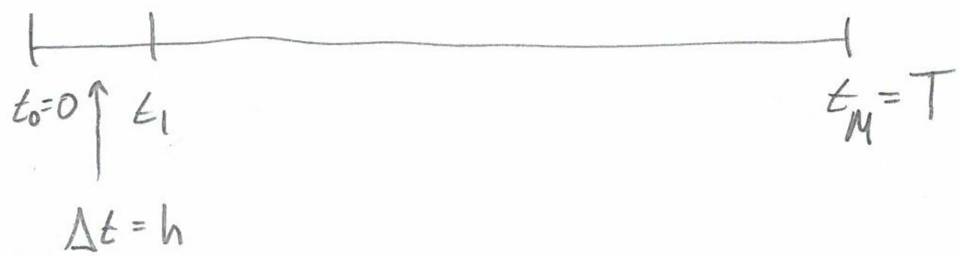
$$v' = -cv - kx$$

Euler's Method:

$$u' = f(t,u)$$

$\uparrow$

need to replace with a discrete approx.



$$u(t_i+h) = u(t_i) + u'(t_i)h + u''(\eta_i)\frac{h^2}{2}$$

$\downarrow$

$\cancel{h\tau_i = h u''(\eta_i)\frac{h}{2}, \quad \eta_i \in [t_i, t_{i+1}]}$

~~known as the local truncation error.~~

$$\text{So} \quad u'(t_i) = \frac{u(t_i+h) - u(t_i)}{h} \underbrace{- u''(\eta_i)\frac{h}{2}}_{\tau_i}$$

τ_i, local truncation error.

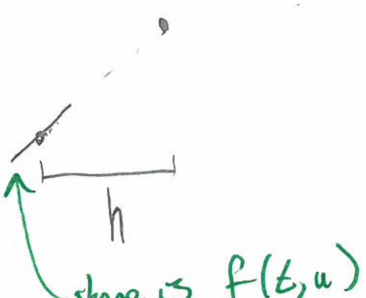
We are going to drop it, under hyp u'' is controlled and h will be made small.

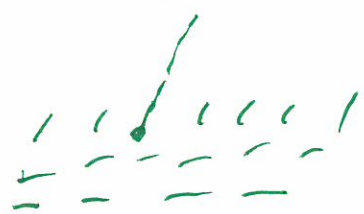
$$\frac{u(t_i+h) - u(t_i)}{h} + \tau_i = f(t_i, u(t_i))$$

We'll look for an approximation $u_i \approx u(t_i)$

$$\frac{u_{i+1} - u_i}{h} = f(t_i, u_i)$$

$$u_{i+1} = u_i + h\, f(t_i, u_i)$$





This then yields a scheme for finding iterates: given u_0, you can compute u_1 and then u_2, etc. This is the (forward) Euler method.

Given

$$u_{i+1} = u_i + hf(t_i, u_i) \qquad \text{we write}$$

$$\underbrace{\frac{u_{i+1} - u_i}{h} - f(t_i, u_i)}_{\text{has units of } u'} = 0$$

Substitute true solution in here

$$\frac{u(t_i + h) - u(t_i)}{h} - f(t_i, u_i) = \frac{u(t_i) + u'(t_i)h + \frac{u''(\eta_i)}{2}h^2 - u(t_i)}{h} - f(t_i, u_i)$$

$$= \frac{u''(\eta_i)h}{2}$$

$$= -\tau_i$$

((Convention varies on the sign here; this is your book's convention.

We would like $\tau_i \to 0$ as $h \to 0$; such a method is called <u>consistent</u>.

I.e. to determine ~~the finite difference scheme is~~ ~~the~~ local truncation error, for a ~~a~~ finite diff approx of $u' = f(t,u)$,

write as ~~[illegible]~~

the discrete form of $u' - f(t,u)$, ~~and~~ substitute in the true solution, and determine the resulting expression $-\tau$. $\tau \to 0$ as $h \to 0$ is consistency.

Following the text, we'll apply it to the logistic equation

$$u' = 10\,u\,(1-u)$$

$$u(0) = 0.01$$

$$\frac{du}{u(1-u)} = 10\,dt$$

$$\frac{1}{u} + \frac{1}{1-u}$$

$$\ln(u) - \ln(1-u) = 10\,dt$$

$$\frac{u}{1-u} = Ce^{10t}$$

which has an exact solution $u = \dfrac{1}{1+9e^{-10t}}$

$$u = (1-u)A$$

$$u(1+A) = A$$

$$u = \frac{A}{1+A}$$

$$= \frac{1}{1+1/A}$$

$$\|E\|_\infty = c\,h^p$$

$$\log(\|E\|_\infty) = \log(c) + p\cdot\log(h)$$

And $h = \dfrac{T}{M}$ so $\log(h) = \log(T) - \log(M)$

$$\log(\|E\|_\infty) = \left[\log(c) - p\log(T)\right] - p\log(M)$$