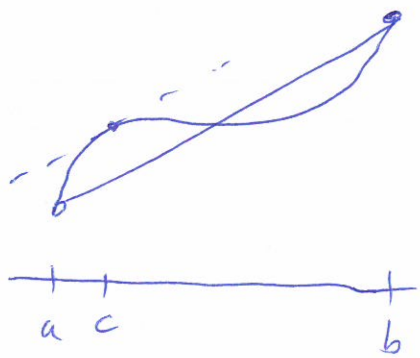
1/16/19

Taylor's Theorem

Two big theorems of Calculus I

1) Mean Value Theorem:



Given $f:[a,b] \to \mathbb{R}$, continuous, differentiable on (a,b), there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$

2) FTC Given $F:[a,b] \to \mathbb{R}$, continuous with a continuous derivative,

$$F(b) - F(a) = \int_a^b F'(s)\,ds$$

These are two facets of the same principle.

$b = a + h$

$b - a = h$

$f(a+h) = f(a) + f'(c) \cdot h$

$$f(a+h) = f(a) + \int_a^{a+h} f'(s)\,ds$$
$$= f(a) + h \cdot \underbrace{\left[\frac{1}{h}\int_a^{a+h} f'(s)\,ds\right]}$$

average value of $f'(s)$ on $[a, a+h]$,

somewhere between max and min of f'.

So is equal to $f'(c)$ for some

c in $[a, a+h]$.

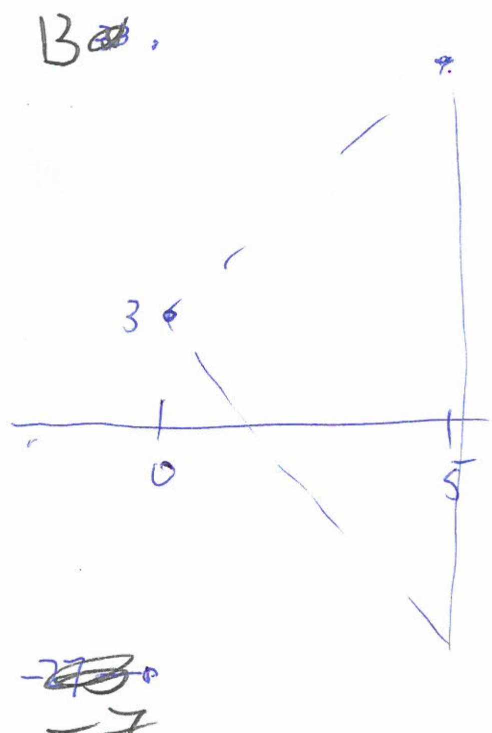
I.e., if f is continuously diff on $[a,b]$, FTC $\Rightarrow$ MVT.

[Emphasis on $h > 0$, but in fact, this is immaterial]

E.g. Suppose $f(0) = 3$ and $|f'(x)| \le 2$ on $[0, 5]$.

How big can $|f(5) - f(0)|$ be?

$$|f(5) - f(0)| = |f'(c)| \cdot 5 \le 2 \cdot 5 = 10.$$



value is in here.

If you approximate $f(5)$ with $f(0)$,

$|f'(c)| \cdot 5$ tells you the

size of your mistake.

Think of it as an error term.

These are the baby versions of Taylor's theorem.

Here's the next edition:

1) $$f(a+h) = f(a) + f'(a)h + \overbrace{\frac{1}{2} f''(c) h^2}^{\text{remainder term. } R_D}$$

for some c in $[a, a+h]$ if f' is continuous on $[a, a+h]$

and f'' exists on $(a, a+h)$.

2) $$f(a+h) = f(a) + f'(a)h + \int_a^{a+h} (a+h-s) f''(s)\, ds \quad \text{if } f \text{ is } C^2.$$

$$\int_a^{a+h} f'(s)\, ds = \int_a^{a+h} -\frac{d}{ds}(a+h-s) f'(s)\, ds$$

$$= \int_a^{a+h} -\frac{d}{ds}\left[(a+h-s) f'(s)\right] ds + \int_a^{a+h} (a+h-s) f''(s)\, ds$$

$$= h f'(a) + \int_a^{a+h} (a+h-s) f''(s)\, ds$$

$$f(a+h) = f(a) + f'(a)h + \underbrace{\int_a^{a+h} (a+h-s) f''(s)\, ds}_{R_I}$$

Moreover,

If $m \le f''(s) \le M$ on the interval

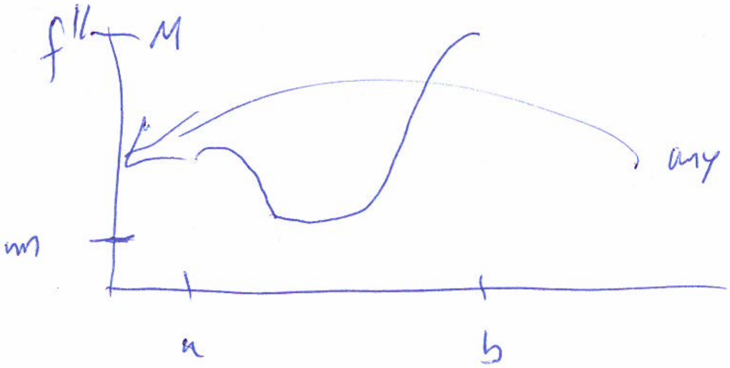
$$m(a+h-s) \le (a+h-s)\, f''(s) \le M\,(a+h-s)$$

$$\int_a^{a+h} (a+h-s)\,ds = -\frac{(a+h-s)^2}{2}\Big|_a^{a+h}$$

$$= \frac{h^2}{2}$$

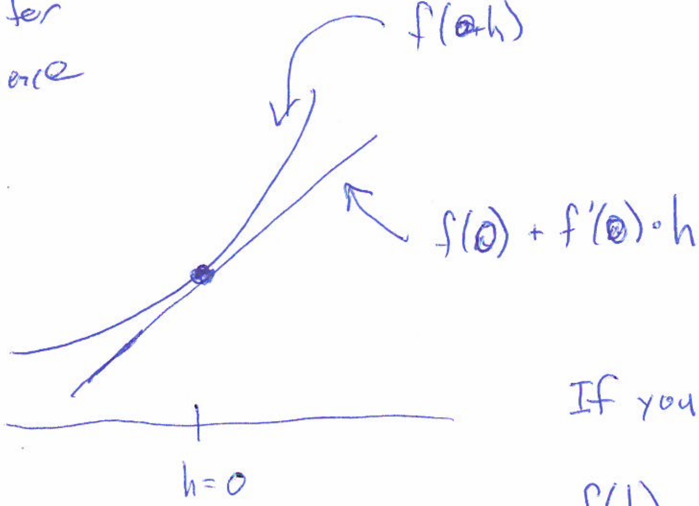
So $\frac{h^2}{2} m \le R_I \le \frac{h^2}{2} M$

$$R_I = \frac{h^2}{2} f''(c) \text{ for some } c \text{ in the interval}$$



any number between is an $f''(c)$

$a = 0$ for convenience



If you approximate

$f(h)$ with $f(0)+f'(0)\cdot h$

your mistake is $\dfrac{f''(c)h^2}{2}$

and if you can estimate $f''(c)$

(i.e you know a bound for it)

then the error is $O(h^2)$

$$\left| f(h) - \left(f(0) + f'(0)\cdot h\right)\right|$$

In general, if $f(x)$ is $k+1$ times continuously diff on $[a, a+h]$

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)h^2}{2} + \cdots + \frac{f^{(k)}(a)h^k}{k!} + \frac{1}{k!}\int_a^{a+h} (a+h-s)^k f^{(k+1)}(s)\,ds$$

Proof is by induction. The base case is $k=0$, FTC.

If the formula holds for some k then

$$\int_a^{a+h} (a+h-s)^k f^{(k+1)}(s)\,ds = -\int_a^{a+h} \left[\frac{d}{ds}\frac{(a+h-s)^{k+1}}{k+1}\right] f^{(k+1)}(s)\,ds$$

$$= -\int_a^{a+h} \frac{d}{ds}\left[\frac{(a+h-s)^{k+1}}{k+1} f^{(k+1)}(s)\right]$$

$$+ \int_0^{a+h} \frac{(a+h-s)^{k+1}}{k+1} f^{(k+2)}(s)\,ds$$

$$= \frac{h^{k+1}}{k+1} f^{(k+1)}(a) + \int_0^{a+h} \frac{(a+h-s)^{k+1}}{k+1} f^{(k+2)}(s)\,ds$$

So

$$f(a+h) = f(a) + f'(a)h + \cdots + \frac{f^{(k)}(a)h^k}{k!} + \frac{h^{k+1}}{k!(k+1)} f^{(k+1)}(a) + \int_a^{a+h} \frac{(a+h-s)^{k+1}}{(k+1)!} f^{(k+2)}(s)\,ds$$

Moreover,

$$\int_a^{a+h} \frac{(a+h-s)^k}{k!}\,ds = -\frac{(a+h-s)^{k+1}}{(k+1)!}\Bigg|_a^{a+h}$$

$$= \frac{h^{k+1}}{k+1!}$$

By a similar argument to the prior,

$$R_I = \underbrace{\frac{h^{k+1}}{(k+1)!} f^{(k+1)}(c)}_{R_D} \text{ for some } c \text{ in the interval.}$$

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)h^2}{2!} + \cdots + \frac{f^{(n)}(a)\,h^n}{n!} + R_D$$

E.g. Estimate the error in approximating $\ln(x)$ by its 10^{th} order Taylor polynomial at 1 on $[0, 1.5]$

Derivatives: $\ln(x), +\frac{1}{x}, -\frac{1}{x^2}, \frac{2}{x^3}, \frac{-3!}{x^4}, \ldots, -\frac{9!}{x^{10}}$

Taylor polynomial $0, 1, -1, 2!, -3!, \ldots, -9!$

$$0 + h - \frac{h^2}{2!} + \frac{2!\,h^3}{3!} - \cdots - \frac{9!\,h^{10}}{10!}$$

$$\ln(1+h) = 0 + h - \frac{h^2}{2} + \frac{h^3}{3} \cdots - \frac{h^{10}}{10} + R$$

Remainder $R = \frac{\ln^{(11)}(c)}{11!} h^{11}$

$\ln^{(11)}(c) = \frac{10!}{c^{11}}$ so $R = \left(\frac{h}{c}\right)^{11} \cdot \frac{1}{11}$ $\quad |h| < \frac{1}{2}$

$\left|\frac{1}{c}\right| < 1$

$$|R| < \left(\frac{1}{2}\right)^{11} \frac{1}{11} = 0.000044$$