1/14/19

The heat equation (a cautionary tale)

$u(x,t)$

$$u_t = u_{xx} \quad \text{(heat equation)}.$$

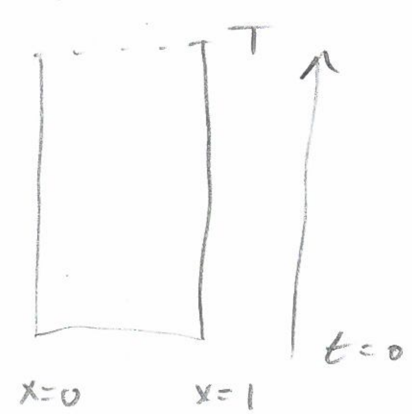
we'll solve for $0 \leq x \leq 1$

$0 \leq t \leq T$ ($T = 0.1$ in our examples).

For intuition, can think of u as temp, t as time.

~~Boundary conditions~~

Domain

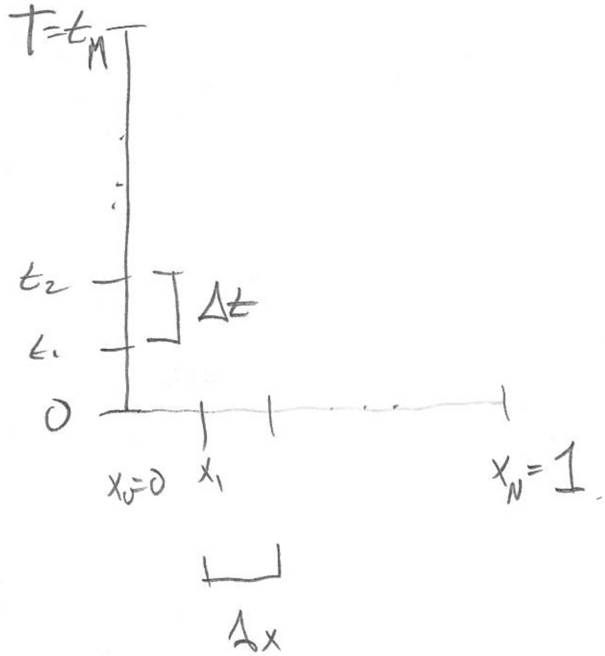


initial condition: $u(x,0) = u_0$ (initial temp distribution)

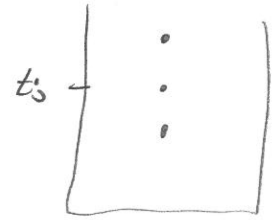
boundary conditions: $u(0,t) = 0$

$u(1,t) = 0$

Strategy:

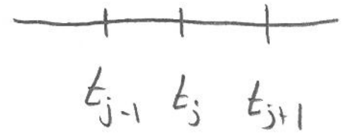


$$\Delta t = \frac{T}{M}$$



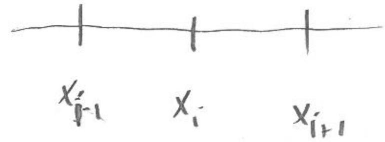
$$\Delta x = \frac{1}{N}$$

$$u_t(x_i, t_j) \approx \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\Delta t}$$



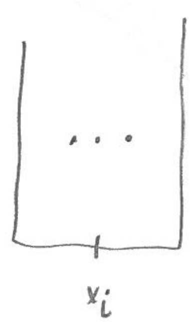
$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

$$u_x(x_i, t_j) \approx \frac{u(x_{i+1}, t_j) - u(x_i, t_j)}{\Delta x}$$



and

$$u_x(x_i, t_j) \approx \frac{u(x_i, t_j) - u(x_{i-1}, t_j)}{\Delta x}$$



D_+ , D_-

Combine and $u_{xx} \approx \dfrac{u_{i+1,j} - u_{i,j} - (u_{i,j} - u_{i-1,j})}{\Delta x^2}$

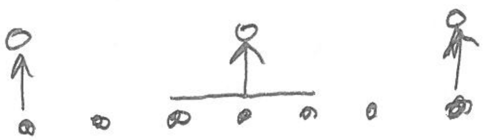
$$= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

Approximate heat equation

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

$$u_{i,j+1} = u_{i,j} + \frac{\Delta t}{\Delta x^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

except $u_{0,j} = 0$
$u_{N,j} = 0$ always.



Each value $u_{i,j+1}$ can be determined from the values $u_{i,j}$.

\# of time steps

I've written heat_basic(u_0, T, M)

$u_{i,0}$, vector of values at $x_0, x_1, \dots, x_N$

$[u_0, u_1, \dots, u_M]$ ← returns a list of these.

Important: let's verify that our code seems to work.

exact solutions: $u(x,t) = \sin(k\pi x)\, e^{-k^2\pi^2 t}$ $\quad k = 1,2,3,\dots$

$$u_{xx} = -(k\pi)^2 \sin(k\pi x)\, e^{-k^2\pi^2 t}$$

$$u_t = -k^2\pi^2 \sin(k\pi x)\, e^{-k^2\pi^2 t}$$

$u_t = u_{xx}$ ✓

$u(0,t) = \sin(0) = 0$ ✓

$u(1,t) = \sin(k\pi)\, e^{-k^2\pi^2 t} = 0$ ✓