

1. Consider the heat equation  $u_t = \kappa u_{xx}$  for  $\kappa > 0$ ,  $x \in [0, 1]$ , and Dirichlet boundary conditions  $u(0, t) = 0$  and  $u(1, t) = 0$ . Suppose we have initial condition  $u(x, 0) = \sin(5\pi x)$ .
  - a) Find an exact solution to this problem.
  - b) Implement the backward Euler (BE) method to solve this heat equation problem. Specifically, use diffusivity  $\kappa = 1/20$  and final time  $T = 0.1$ . Note that you do not need to use Newton's method to solve the implicit equation, which is a linear system, but you should use sparse storage and an efficient linear solver (backslash in MATLAB will work).
  - c) Suppose the timestep  $k$  and the space step  $h$  are related by  $k = 2h$ . What do you expect for the convergence rate  $O(h^p)$ ? Then measure it by using the exact solution from a), at the final time, and the infinity norm  $\|\cdot\|_\infty$ , and  $h = 0.05, 0.02, 0.01, 0.005, 0.002, 0.001$ . Make a log-log convergence plot of  $h$  versus the error.
  - d) Repeat parts b) and c) but with the trapezoidal rule instead of BE. (That is, implement and measure the convergence rate of Crank-Nicolson, with everything else the same.)

2. Consider the PDE

$$u_t = \partial_x(p(x)u_x)$$

where  $p(x)$  is a given function. We wish to solve the PDE on the region  $0 \leq x \leq 1$ ,  $0 \leq t \leq T$  with  $u = 0$  at  $x = 0, 1$ . We will apply the following finite difference scheme to it:

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2} [(u_{i+1,j} - u_{i,j})p_{i+\frac{1}{2}} - (u_{i,j} - u_{i-1,j})p_{i-\frac{1}{2}}]$$

where  $p_{i\pm\frac{1}{2}} = p(x_i \pm h/2)$ .

- a) Estimate the local truncation error in terms of powers of  $h$  and  $k$  and in terms of derivatives of  $u$  and derivatives of  $p$ . I'm looking for an answer akin to the estimate we derived for the heat equation of the form

$$|\tau| \leq \max |u_{xxxx}| \left[ \frac{k}{2} + \frac{h^2}{h} \right]$$

that we derived for the heat equation with no forcing term.

- b) Show that the method is convergent, assuming  $0 < p(x)k < h^2/2$ . You will want to revisit the proof from class that the explicit method for the standard heat equation is convergent.

3.

a) Let

$$A = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Compute  $\|A\|_1$  and  $\|A\|_\infty$ .

b) Estimate  $\|A\|_2$  as follows. Computer generate a figure containing the boundary of  $A(B_1)$ , where  $B_1$  is the Euclidean ball of radius 1. Then use the figure to estimate the norm.

c) Suppose  $A$  is an  $n \times n$  matrix, and choose  $p \in [1, \infty]$ . Show that  $\|A\|_p = 0$  if and only if  $A$  is the 0 matrix.

d) For vectors in  $\mathbb{R}^n$ , it is known that  $\|x + y\|_p \leq \|x\|_p + \|y\|_p$  for any  $p \in [1, \infty]$ . This is the triangle inequality, and you need not prove it. But using this fact, show that the triangle inequality also holds for matrix norms  $\|\cdot\|_p$  for  $p$  in the same range.

4. Text, problem 3.7