1. Andrew gave us the following definition. A space rotation is a linear map R from $\mathbb{R}^3$ to itself such that there exists perpendicular unit vectors u and v and a real number θ such that

$$\begin{aligned} R(u) &= u \\ R(v) &= \cos(\theta)v + \sin(\theta)w \\ R(w) &= -\sin(\theta)v + \cos(\theta)w \end{aligned} \tag{1}$$

where $w = u \times v$.

a) Show that for every axis u and angle θ, there exists a corresponding rotation. Do this by showing that conjugation by a particular unit quaternion suffices. Show also that all rotations with this axis and angle are the same maps. That is, the role of v in the definition is a bit immaterial.

b) Show that the only unit quaternions t such that

$$t^{-1}qt = q$$

for all imaginary unit quaternions are $t = 1$ and $t = -1$.

c) Conclude that for every axis u and angle θ, there is a unique pair t and $-t$ of unit quaternions such that conjugation by t yeilds the corresponding rotation.

2. Text: 1.4.2

3. Text: 1.5.1, 1.5.2

4. For $p, q \in \mathbb{H}$, we define $\langle p, q \rangle = \mathbf{Re}(p\overline{q})$, where $\mathbf{Re}q$ is the real part of q.

a) Show that if $p \in \mathbb{H}$, then $\langle p, p \rangle = |p|^2$.

b) Using the facts that $\overline{ab} = \overline{b}\,\overline{a}$ and $\mathbf{Re}(\overline{a}) = \mathbf{Re}(a)$ for all $a, b \in \mathbb{H}$, show that $\langle p, q \rangle = \langle q, p \rangle$.

c) For $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, let $Q(x) = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ be its representation as a quaternion. Show that for $x, y \in \mathbb{R}^4$, $x \cdot y = \langle Q(x), Q(y) \rangle$.

d) We say that p and q are orthogonal if $\langle p, q \rangle = 0$. Show that $p \in \mathbb{H}$ is orthogonal to 1 if and only if p is pure imaginary.

e) Suppose $s \in \mathbb{H}$. Show that for all $p, q \in \mathbb{H}$, $\langle ps, qs \rangle = |s|^2 \langle p, q \rangle$.

f) Show that for all $p, q \in \mathbb{H}$, $\langle \overline{p}, \overline{q} \rangle = \langle p, q \rangle$.

g) Suppose $s \in \mathbb{H}$. Show that for all $p, q \in \mathbb{H}$, $\langle sp, sq \rangle = |s|^2 \langle p, q \rangle$.

h) Suppose $u \in \mathbb{H}$ and $u \neq 0$. Show that conjugation by u takes the real quaternions to themselves.

i) Conclude quickly that if $u \in \mathbb{H}$ and $u \neq 0$, then conjugation by u takes the imaginary quaternions to themselves.