In this document we establish the following facts about $\mathbb{R}^n$:

1. Any two norms are equivalent.
2. The closed unit ball with respect to any norm is compact.
3. It is complete, with respect to any norm.

We then show that these same facts are true for any finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$.

For $\mathbb{R}^n$, it will be convenient to work with the infinity norm,

$$||x||_\infty = \max_{i=1}^n |x_i|.$$

We first establish facts 1 and 2 for $||\cdot||_\infty$.

Lemma 1: The $||\cdot||_\infty$ closed unit ball $\{x \in \mathbb{R}^n : ||x||_\infty \le 1\}$ in $\mathbb{R}^n$ is compact with respect to $||\cdot||_\infty$.

Proof. Let $\{x_j\}_j$ be a sequence in $\mathbb{R}^n$ with $||x_j||_\infty \le 1$ for each j and let us write $x_j = (x_j(1), \dots, x_j(n))$. Observe that each $|x_j(1)| \le 1$ and hence there is a subsequence $\{x_{j_k}\}_k$ that converges to some $x(1) \in \mathbb{R}$ with $|x(1)| \le 1$ also. Applying this argument to $\{x_{j_k}\}_k$ we find a sub-subsequence with $x_{j_{k_\ell}}(2) \to x(2)$ for some $x(2)$ with $|x(2)| \le 1$. Further repeating this argument we can extract a final subsequence, call it $\{y_k\}$, where for each $j = 1, \dots, n$, $y_k(j) \to x(j)$ for some $x(j)$ with $|x(j)| \le 1$.

Setting $x = (x(1), \dots, x(n))$, it is clear that $||x||_\infty \le 1$. We claim that $y_k \to x$ with respect to $||\cdot||_\infty$. Indeed, let $\epsilon > 0$. For each j pick K_j such that if $k \ge K_j$, then

$$|y_k(j) - x(j)| < \epsilon.$$

Then if $k \ge K = \max(K_1, \dots, K_n)$ then

$$||y_k - x||_\infty = \max_{j=1}^n |y_k(j) - x(j)| < \epsilon.$$

□

There is nothing special about unit balls compared to balls of an arbitrary radius:

Exercise 1: Use the previous result to show that for any $R \ge 0$, $B_R = \{x \in \mathbb{R}^n : ||x|| \le R\}$ is compact.

The following three exercises are straightforward applications of the definitions for metric spaces.

Exercise 2: Show that a Cauchy sequence in a metric space is bounded.

Exercise 3: Show that a Cauchy sequence in a metric space with a convergent subsequence actually converges.

Lemma 2: The space $\mathbb{R}^n$ is complete with respect to $\|\cdot\|_\infty$.

Proof. Let $\{x_n\}$ be Cauchy with respect to $\|\cdot\|_\infty$. Then the sequence is bounded and there is some R such that $\|x_n\|_\infty \le R$ for all n. The closed ball of radius R is compact, and thus the sequence admits a subsequence converging to some x with $\|x\|_\infty \le R$. But Cauchy sequences with convergent subsequences converge and thus $x_n \to x$. □

We now show that any norm on $\mathbb{R}^n$ is equivalent to $\|\cdot\|_\infty$. This establishes fact 1, since equivalence of norms is transitive. The key ingredients of the argument are the compactness of the $\|\cdot\|_\infty$ unit ball along with the following.

Lemma 1: Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then it is continuous with respect to $\|\cdot\|_\infty$.

Proof. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$ and let $M = \max_{i=1}^n \|e_i\|$. Applying the triangle inequality and Theorem 2.11(a) we find that for all $x, y \in \mathbb{R}^n$

$$\begin{aligned} \big|\|x\| - \|y\|\big| &\le \|x - y\| \\ &= \left\| \sum_{i=1}^n (x_i - y_i) e_i \right\| \\ &\le \max_{i=1}^n |x_i - y_i| M \\ &= M\|x - y\|_\infty. \end{aligned}$$

Thus, given $\epsilon > 0$, if $\|x - y\|_\infty < \epsilon/M$, then $|\|x\| - \|y\|| < \epsilon$, which shows that $\|\cdot\|$ is uniformly continuous with repsect to $\|\cdot\|_\infty$. □

Proposition 3: If $\|\cdot\|$ is any norm on $\mathbb{R}^n$, it is equivalent to $\|\cdot\|_\infty$.

Proof. Let $S = \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$. Then S is a closed subset of the closed unit ball and is hence compact. Since $\|\cdot\|$ is a continuous function, and since S is compact, there exist x_- and x_+ in S such that $\|x_-\| \le \|x\| \le \|x_+\|$ for all $x \in S$. Let $m = \|x_-\|$ and let $M = \|x_+\|$, and observe that $m > 0$ since $0 \notin S$. Given any $x \in \mathbb{R}^n$ with $x \neq 0$, it follows that

$$m \le \left\| \frac{x}{\|x\|_\infty} \right\| \le M$$

and hence

$$m\|x\|_\infty \le \|x\| \le M\|x\|_\infty.$$

This same inequality holds trivially if $x = 0$ as well. □

Recall the following facts about equivalent norms proved in class:

1. If a sequence converges with respect to one of the norms, it converges to the same limit with respect to the other norm.

2. If a sequence is Cauchy with respect to one norm, then it is with respect to the other norm.
3. If a space is complete with respect to one norm, then it is complete respect to the other norm.

With these facts in hand, we obtain the following two results.

Proposition 4: Given a norm $||\cdot||$ on $\mathbb{R}^n$, the closed unit ball $B = \{x \in \mathbb{R}^n : ||x|| \le 1\}$ is compact (with respect to $||\cdot||$).

Proof. Suppose $\{x_n\}_n$ is a sequence in $\mathbb{R}^n$ with $||x_n|| \le 1$ for each n. By equivalence of norms, there exists $R > 0$ such that $||x||_\infty \le R||x||$ for all $x \in \mathbb{R}^n$. Since the $||\cdot||_\infty$ ball of radius R is compact, the sequence $\{x_n\}_n$ admits a subsequence converging to some x. By equivalence of norms, the sequence also converges with respect to $||\cdot||$. Since $||x_n|| \le 1$ for each n, the continuity of the norm implies $||x|| \le 1$ as well. □

Proposition 5: The space $\mathbb{R}^n$ is complete with respect to any norm.

Proof. The space $\mathbb{R}^n$ is complete with respect to $||\cdot||_\infty$, and any norm on $\mathbb{R}^n$ is equivalent to $||\cdot||_\infty$. □

We now turn to an arbitrary finite-dimensional real vector space X and we pick, once and for all, a basis $E_1, \ldots E_n$ for it.

Exercise 4: Let $||\cdot||$ be a norm on X. Given $p \in \mathbb{R}^n$, $p = (p(1), \ldots, p(n))$ define

$$||p||' = ||p(1)E_1 + \cdots p(n)E_n||.$$

Show that $||\cdot||'$ is a norm on $\mathbb{R}^n$. You will need to use the fact that the vectors $E_1, \ldots, E_n$ are linearly independent to show that if $||p||' = 0$ then $p = 0$.

The equivalence of norms on X is now a simple corollary of the equivalence of norms on $\mathbb{R}^n$.

Proposition 6: Any two norms on a finite-dimensional vector space over $\mathbb{R}$ are equivalent.

Proof. Let $||\cdot||_1$ and $||\cdot||_2$ be two norms on X and let $||\cdot||_1'$ and $||\cdot||_2'$ be the associated norms on $\mathbb{R}^n$ via the construction of Exercise 4. There exist constants m and M such that

$$m||p||_1' \le ||p||_2' \le M||p||_1'$$

for all $p \in \mathbb{R}^n$. Given $x \in X$ we can write $x = x(1)E_1 + \cdots x(n)E_n$ for certain coefficients $x(j)$. Setting $p = (x_(1), \ldots, x(k))$ we see that $||p||_1' = ||x||_1$ and $||p||_2' = ||x||_2$. Thus

$$m||x||_1 \le ||x||_2 \le M||x||_1$$

□

Proposition 7: The closed unit ball with respect to any norm on a finite-dimensional vector space is compact.

Proof. Let $\{x_j\}$ be a sequence in X with $\|x_j\| \le 1$ for each j, and write $x_j = x_j(1)E_1 + \cdots + x_j(n)E_n$. Defining $p_j \in \mathbb{R}^n$ by $p_j = (x_j(1), \ldots, x_j(n))$ we find $\|p_j\|' \le 1$, where $\|\cdot\|'$ is the norm on $\mathbb{R}^n$ via the construction of Exercise 4. Thus $p_{j_k} \to p = (p(1), \ldots, p(n))$ for some subsequence and some limit p. Letting $x = p(1)E_1 + \cdots + p(n)E_n$ and unwinding definitions we find $x_{j_k} \to x$ as well. By continuity of the norm, $\|x\| \le 1$ and we are done. □

Exercise 5: Follow the strategy of Lemma 2 to show that any finite-dimensional normed space is complete.

For vector spaces over $\mathbb{C}$, we use the observation that any vector space over $\mathbb{C}$ is simultaneously a vector space over $\mathbb{R}$, where scalar multiplication is defined the same way, but we only permit multiplication by real numbers.

Exercise 6: If X is a complex vector space with norm $\|\cdot\|$, show that $\|\cdot\|$ remains a norm when thinking of X as a vector space over $\mathbb{R}$.

Exercise 7: If X is a finite-dimensional complex vector space of dimension n, then it is a finite-dimensional real vector space of dimension $2n$. Hint: If $x_1, \ldots, x_n$ form a basis for X over $\mathbb{C}$, show $x_1, \ldots, x_n$ together with $ix_1, \ldots, ix_n$ form a basis for X over $\mathbb{R}$.

Exercise 8: With the previous observations in hand, establish the following (with little labor).

1. Any two norms on a finite-dimensional complex vector space are equivalent.
2. The closed unit ball in a finite-dimensional complex vector space is compact.
3. A finite-dimensional normed complex vector space is complete.