Preamble: There is a total of **56** points on this exam; not every problem is equally weighted. Please write out your proofs and your statements of theorems and definitions as clearly as you can. In grading, however, some consideration will be made for the exam's time constraints. Good luck!

1. [14 points]

Define the following.

a) A Banach space.

b) An inner product space.

c) The dual space of a normed space.

d) The norm of a linear map $T : X \to Y$, where X and Y are normed linear spaces.

e) A Schauder basis for a Banach space.

f) An orthonormal basis for a Hilbert space.

g) A bountiful subset of a metric space.

2. [16 points]

State the following

a) Hölder's inequality for ℓ_q.

b) Bessel's inequality.

c) Cauchy-Schwartz inequality.

d) Parallelogram law.

e) Baire Category Theorem (any variation is fine).

f) Banach-Steinhaus Theorem.

g) Riesz Representation Theorem.

h) An important characterization of finite dimensional normed spaces.

3. [16 points]

Let X be a normed space and Y be a Banach space. Suppose (T_n) is a bounded sequence in $\mathcal{B}(X, Y)$ and that A is a subset of X such that the sequence (T_n) converges pointwise on A. Prove that the sequence also converges pointwise on $\overline{A}$.

4. [10 points]

Let W be a subspace of an inner-product space X. Suppose $x \in X$ and that there exists a closest point $z \in W$ to x. Show that $x - z \in W^{\perp}$.