Recall that $\hat{L}^1[a,b]$ is the set of equivalence classes of Riemannian integrable functions on $[a,b]$, where $f \sim g$ if

$$\int_a^b |f-g| = 0.$$

1. Show how to impose a suitable vector space structure on $\hat{L}^1[a,b]$. For the purposes of this exercise you must define

$$[f]+[g]$$

and

$$\alpha[f]$$

and show that this definition is well-defined. Then, rather than showing that all the vector space axioms hold, just show these two:

1. There exists an element $Z \in \hat{L}^1[a,b]$ with $Z+F=F$ for all $F \in L^1[a,b]$.
2. For all $\alpha \in \mathbb{R}$ and $F, G \in L^1[a,b]$, $\alpha(F+G) = \alpha F + \alpha G$.

2. Show that $\hat{L}^1[a,b]$

$$||F|| = \int_a^b f$$

where f is any function with $[f]=F$ is a well-defined norm on $\hat{L}^1[a,b]$.

3. Show that $\hat{L}^1[a,b]$ is not complete. Here's one approach on $[0,1]$. Consider the functions

$$f_n(x) = \begin{cases} x^{-1/2} : x \ge 1/n \\ 0 : \text{otherwise.} \end{cases}$$

Show that the sequence $[f_n]$ is Cauchy in $\hat{L}^1[a,b]$. And then show that if f is a Rieman integrable function that $f_n \not\to f$. Hint: if f is Riemann integrable, there is a number M with $|f| \le M$. Now show that $||[f]-[f_n]|| \ge 1/(2M)$ if n is sufficiently large.

4. Using the rules presented in class for $L^1[a,b]$ show that if f is a representative of both F and G in $L^1[a,b]$ then $F=G$.

5. TBA