Please see the rules on the second page.

1. Suppose X and Y are Banach space and $\{T_n\}$ is a sequence of invertible operators in $B(X,Y)$ converging to some operator T. Suppose moreover, there is a constant $C > 0$ such that $||T_n(x)|| \geq C||x||$ for all $x \in X$ and $n \in \mathbb{N}$. Show that T has a continuous inverse.

2. Suppose $\{x_n\}$ is a sequence that converges weakly to x in a Banach space X. Let $T \in B(X,Y)$ be a compact operator. Show $T(x_n) \to T(x)$.

3. Suppose $T \in B(X)$ for some Banach space and

$$S = \prod_{k=1}^{n} (T - \mu_i I)$$

for certain complex numbers μ_i. Show that S is invertible if and only if each $T - \mu_i I$ is.

4. Let $f \in C[0,1]$ and define $T_f : C[0,1] \to C[0,1]$ by $T_f(g) = fg$. Compute $\sigma(T_f)$ and $\sigma_p(T_f)$.

5. If I is a closed, bounded interval in $\mathbb{R}$, we define $H^1(I)$ to be the closure of the smooth functions on I with respect to the norm

$$||u|| = ||u||_2 + ||u'||_2.$$

a) Show that for smooth functions, the identity map from $H^1(I)$ to $C[0,1]$ is continuous. Hint: The Fundamental Theorem of Calculus will be handy.

b) The previous subproblem shows that elements of $H^1(I)$ can be identified with continuous functions. But elements of $H^1(I)$ need not be smooth. Indeed, show that $f(x) = |x|$ belongs to $H^1([-1,1])$.

6. A set $\{\phi_j\}$ in a Hilbert space X is called a Riesz basis if there is a continuous linear isomorphism $T : X \to X$ such that $\{\phi_j\}$ is the image under T of an orthonormal basis.

A set $\{\psi_j\}$ is a Schauder basis if it is linearly independent and if $\overline{\text{Span}\{\psi_j\}} = X$.

a) Give an example of a Schauder basis for ℓ_2 that is not a Riesz basis.

b) Suppose $\{\phi_j\}$ is a Riesz basis. Show that $\sum_{j=1}^{\infty} c_j \phi_j$ converges if and only if $\{c_j\} \in \ell_2$.

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- You may not discuss the exam with anyone else until after the due date/time.
- You are permitted to reference our course text, but no other references.
- Each problem is weighted equally.
- The due date/time is absolutely firm.
- We will schedule a hints session at a time TBA.