Pf of OMT:

Suppose $T \in B(X,Y)$ is surjective and X and Y are Banach spaces. Then T is open.

Pf: Let $\mathcal{B}_X = B_1(0,X)$.

Let $\mathcal{G}_Y = \overline{T(\mathcal{B}_X)}$.

By linearity $r\mathcal{G}_Y = r\,\overline{T(\mathcal{B}_X)}$

$$= \overline{rT(\mathcal{B}_X)}$$

$$= \overline{T(r\mathcal{B}_X)}.$$

Hence $\bigcup_{n\in\mathbb{N}} n\mathcal{G}_Y = \bigcup_{n\in\mathbb{N}} T(n\mathcal{B}_X) = T\left(\bigcup n\mathcal{B}_X\right) = T(X) = Y.$

Since Y is complete, the Baire Category Theorem implies some $n\mathcal{G}_Y$ contains an open ball and hence also

closed set

$\mathcal{G}_T = \overline{T(\mathcal{B}_X)}$.

Now $T(B_X)$ is symmetric about 0 and convex, and hence so is $\mathcal{G}_Y = \overline{T(B_X)}$.

Observe $\mathcal{G}_Y \supseteq B_r(y_0, Y)$ and

$$\mathcal{G}_Y \supseteq B_r(-y_0, Y) \text{ by symmetry about } 0.$$

But then if $\|w\| < r$, $-y_0 + w$ and $y_0 + w \in \mathcal{G}_Y$.

By convexity $w = \frac{1}{2}(y_0 + w) + \frac{1}{2}(-y_0 + w) \in \mathcal{G}_Y$ as well.

So $B_r(0, Y) \subseteq \mathcal{G}_Y = \overline{T(B_1(0,X))}$.

By the technical lemma, $B_{\frac{r}{2}}(0, Y) \subseteq T(B_1(0, X))$.

But then for any $\varepsilon > 0$, $T(B_\varepsilon(0, X)) \supseteq B_{\frac{r\varepsilon}{2}}(0, Y)$.

Now let $U \subseteq X$ be open and let $y \in T(U)$.

Pick $x \in U$ with $Tx = y$. There exists $\varepsilon > 0$ with $B_\varepsilon(x, X) \subseteq U$.

But then
$$\begin{aligned} T(B_\varepsilon(x, X)) &= T(x) + T(B_\varepsilon(0, X)) \\ &\supseteq T(x) + B_{\frac{r\varepsilon}{2}}(0, Y) \\ &= B_{\frac{r\varepsilon}{2}}(y, Y). \end{aligned}$$
So TU is open.

of BIT

Cor: Suppose $T \in B(X,Y)$ and X, Y are Banach spaces.
Then TFAE

1) T is invertible

2) $T(X)$ is dense in Y and $\exists c$, $\|T(x)\| \geq c\|x\|$ for all $x \in X$.

Pf: If T is invertible $T(X) = Y$ and given any $x \in X$, $x = T^{-1}(y)$ and

$$\|x\| = \|T^{-1}(y)\| \leq \|T^{-1}\| \, \|y\| = \|T^{-1}\| \, \|Tx\|$$

So $c = \|T^{-1}\|^{-1}$ works.

Conversely, suppose $T(X)$ is dense and $\exists c$, $\|T(x)\| \geq c\|x\|$ $\forall x$.

We need only show T is bijective. $\leftarrow$ BLT!

T is injective, for if $T(x) = 0$, $c\|x\| = 0 \Rightarrow x = 0$.

As for surjectivity, given $y \in Y$ find x_n's, $Tx_n \to y$.

Then $\{Tx_n\}$ is Cauchy, as is $\{x_n\}$ as

$$\|x_n - x_m\| \leq c\,\|T(x_n - x_m)\| = c\,\|Tx_n - Tx_m\|.$$

So $x_n \to x$ for some x and $Tx_n \to Tx$.

Cor: If $T \in B(X,Y)$ between Banach spaces then exactly one of the following is true

a) T is invertible

b) $T(X)$ is not dense, or there is a sequence $\{x_n\}$ in X, $\|x_n\|=1$, $\|Tx_n\| \to 0$

(either of b $\Rightarrow$ not invertible)

($\|Tx\| \geq \frac{1}{n}\|x\|$ fails for each n for some $x \neq 0$

So is $x_n \neq 0$, $\|Tx_n\| \leq \frac{1}{n}\|x_n\|$ and

can assume WLOG $\|x_n\|=1$).

Recall $I(f_n)$ $f_n = x^n = \frac{1}{n+1} y^{n+1}$

$\|f_n\|=1$ $\|If_n\|_\infty = \frac{1}{n+1}$

So $I: C[0,1] \to C[0,1]$ can't be invertible.

And since I is injective, the image of I can't be closed: it would be a Banach space and I would be invertible!

Related result:

Closed Graph Theorem

Suppose $T: X \to Y$ is linear and X, Y are Banach spaces. Then T is continuous iff

$$\text{Graph}(T) = \{ (x, Tx) : x \in X \} \text{ is closed in } X \times Y.$$

Note Graph(T) is a subspace $(x, Tx) + (\hat{x}, T\hat{x})$

$$= (x+\hat{x}, Tx + T\hat{x})$$

$$= (x+\hat{x}, T(x+\hat{x})).$$

What's the big deal?

From the def, to show that T is cts, need to show if $x_n \to x$ then $Tx_n \to Tx$

need to show Tx_n <u>converges</u> and

its limit is Tx

But to apply CGT, need to show that the graph is closed.

i.e. if $(x_n, y_n) \in \text{Graph } T$

$(x_n, y_n) \to (x, y)$ then $(x,y) \in \text{Graph } T$.

$$y_n = Tx_n$$

That is if $x_n \to x$

$Tx_n \to y$

then $y = Tx$.

You get to <u>assume</u> Tx_n converges to something.

e.g. If $p \le q$ $\ell^p \subseteq \ell^q$:

$$\sum_{k=1}^{\infty} |x_k|^q = \sum_{k=1}^{N} |x_k|^q + \sum_{k=N+1}^{\infty} |x_k|^q$$

$$\le \sum_{k=1}^{N} |x_k|^q + \sum_{k=N+1}^{\infty} |x_k|^p$$

$\left(|x_k| \le 1 \right.$

$\Rightarrow |x_k|^p \ge |x_k|^q$.

I claim $(\ell^p, \ell^p) \to (\ell^p, \ell^q)$ is cts.

Suppose $(x_n) \in \ell^p$ $\quad x_1 \xrightarrow{\ell^p} x$

and $\quad x_n \xrightarrow{\ell^q} y$.

For each k $\quad x_n(k) \to x(k)$

$x_n(k) \to y(k)$ $\quad$ so $\quad x(k) = y(k)$.

Pf: Suppose T is cts and

$$(x_n, Tx_n) \to (x, y).$$

Then $x_n \to x$ so $Tx_n \to Tx$. Since $Tx_n \to y$, $y = Tx$

and $(x, y) \in \text{Graph } T$.

Suppose Graph T is closed. Then Graph T is a closed subset of a Banach space and is a Banach space.

Define $\pi_X : \text{Graph}(T) \to X$
$\qquad (x, Tx) \to x$

$\pi_Y : X \times Y \to Y$
$\qquad (x, y) \to y$.

Observe π_X, π_Y are cts, linear. Since π_X is bijective,

π_X has a continuous inverse. Now observe $\pi_2 \circ \pi_1^{-1}(x) = \pi_2(x, Tx) = T(x)$

Most mysterious: Banach Steinhaus (pointwise bounded $\Rightarrow$ uniformly bounded)

If $\{T_\alpha\}_{\alpha \in I}$ is a family in $B(X, Y)$ (Banach)

and for each x, $\{T_\alpha(x)\}$ is bounded,

then $\exists$ M, $\|T_\alpha\| \le M$ for all α.

Application: If $T_n \in B(X,Y)$ and for each x, $T_n(x) \to T(x)$, then $T \in B(X,Y)$.

pf

$T_n(x+y) \to T(x+y)$

$T_n(x+y) = T_n x + T_n y \to Tx + Ty$ etc. T is linear.

Now since $T_n x \to Tx$ the operators $\{T_n\}$ are pointwise bounded. So is M, $\|T_n\| \le M$.

But $\|Tx\| = \lim \|T_n x\| \le M\|x\|$. So $\|T\| \le M$.

pointwise convergence implies limit is cts.

(ODD!)

Application:

$f \in C[-\pi,\pi] \qquad f(-\pi) = f(\pi)$

Fourier coeffs $\quad c_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f$

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kx)\, f(x)\, dx$$

$$d_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(kx)\, f(x)\, dx$$

$$f_N = c_0 + \sum_{k=1}^{N}\left[c_k \cos(kx) + d_k \sin(kx)\right]$$

$$\boxed{\begin{array}{c} f \mapsto f_N \\ C[0,1] \to C[0,1] \\ \text{is cts.} \end{array}}$$

We will show later $\quad$ 1) $f_n \to f$ in L^2

But L^2 convergence is pretty weak. Maybe $f_n \to f$ in $C[0,1]$?

We'll also show $\quad f \to f_N \to f_n(0) \quad$ is $O(\log N)$

$$\underbrace{\qquad\qquad\qquad\qquad}_{T_N}$$

$\|T_N\| \to \infty$. So must be an f $\quad \|T_N f\| \to \infty$

(not unif bounded $\Rightarrow$ not pointwise bounded)

Proof of Banach-Steinhaus:

$\{T_\alpha\}_{\alpha \in I}$

Recall the space $F_b(I, Y)$, bounded maps from I to Y.

$\|f\| = \sup_{\alpha \in I} \|f(\alpha)\|_Y$. Is Banach since Y is.

Given $x \in X$ define $f_x \in F_b(I, Y)$ $f_x(\alpha) = T_\alpha(x)$.

The map $x \xmapsto{\phi} f_x$ is evidently linear.

We'll show it is continuous, via CGT.

Suppose $x_n \to x$, $f_{x_n} \to f$ for some $f \in F_b(I, Y)$.

We need to show $f = f_x$. But

$f_{x_n}(\alpha) \longrightarrow f(\alpha)$ since $\|f_{x_n}(\alpha) - f(\alpha)\|_Y \leq \|f_{x_n} - f\|_b$.

So $T_\alpha(x_n) \longrightarrow f(\alpha)$.

But $T_\alpha(x_n) \longrightarrow T_\alpha(x)$ by continuity.

So $f(\alpha) = f_x(\alpha) \quad \forall \alpha$.