Baire Category Theorem

A complete metric space is not a countable union of nonline douse sets

Nochere dense: A doos not contain an apen ball.

E.g. Q with usion metric.

Sivoletors are nonles lorse. (a-E,a+E)NQ is a ball.

Q is a compuble union of its subletons, so it

Let's apply, then we tun to it.

The inverse of a confinuous linear map need not be continuous.

 $(Z, l') \xrightarrow{id} (Z, l^{\infty})$ 1/×1/00 ≤ 1/×1/, so is ots. $B_{u+} = (1, ..., 1, 0, ...)$ satisfies $\|x_n\|_{0} = 1$, $\|x_n\|_{1} = n$ so id^{-1} is not cts. Bonach Iso morphism Thearm Major vesult: If T: X-7Y is continuous, linear, and bijective and if X, Y are Barach spaces, then T is continuous. HW: If X on Tis Barache und T, T' are cts, so is other.

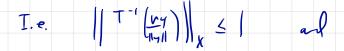
The BIT follows from a more gener), but more technical and less well motivated result: Thun (Open Mupping Theorem) Suppose TEB(Y,Y), Y is a Bonach space, ad T is surjective. Then T is an open map. (I.e. ulerarer UEX soper in X T(U)= { T(x): x = U} is quan in 4. Pf of BIT from OMT The bull B, (O, X) is open so T(B, (O, X)) is open and contrarys O. So there is in 70

 $B_{r}(o, Y) \subseteq B_{2r}(o, Y) \subseteq T(B, (o, X)).$

Suppose YEY Y = 0.

Then $r \neq \epsilon B_r(0) \in T(B_1(0,k))$ $||\gamma||$





 $\|T^{-1}Y\| \leq \frac{1}{2} \|Y\|_{Y}$

This $\|T^{-1}\| \leq \int_{\Gamma} dt = T^{-1}$, is continuous.

Main Technical Lenna Had work. Prop: Suppose T: X-34, X is a Dauch space ad $T(B,(o,x)) \supseteq B(o,y)$ then $T(B_1(0,X)) \ge B_{\underline{r}}(0,Y)$. Pf: Suppose we Briz (0, Y), so Zut Br (0,Y) Find $x_i \in B_i(0, x), || 2w - x_i || < \frac{r}{2}$. Since 23 - 2x + B, (0, 2), we can find x2 + B, (0, 4) 12 w- Zx-tx1 < 5. Continuers inductively, we can find Exiz the B, (0,x), || 2¹/₄ - 2ⁿ/₄ - ... - 2ⁿ/₄ || < ¹/₂. Setting $z_n = \sum_{k=1}^n 2^{-k} x_k$ we find $||w - \overline{z_n}|| < \frac{r}{z^{n+1}}$.

Moneur the series Z B-KE is also cany 50

convers to a limit z and 11211 5 22 11x11 < 1.

Thus ZA -> ZEB, (O, X) and

Tz= lim Tzn= W.

Wormup exercises:

AEX is symmetric about O , & whenever a EA, -acA.

Exercise: If A 13 symmetric about 0, so is A.

Exercise: If ASX is convex, so is A.

PF of OMT: Suppose TG B(X,Y) is sorgective and X and Y are Bouch spuces. Then T is open. $Pf: L_{e} + B_{x} = B_{1}(0, X).$ Let $\mathcal{A}_y = \overline{T(\mathcal{B}_x)}$. By linearity $r \mathcal{L}_Y = r \overline{T(\mathcal{B}_x)}$ $= rT(B_{x})$ $= T(rB_x)$. Have $\bigcup_{n \in \mathbb{N}} A_r = \bigcup_{n \in \mathbb{N}} T(nB_x) = T(\bigcup_{n \in \mathbb{N}} B_x) = T(y) = Y.$ Since Y is complete, the Baine Category Theorem implies Some , n by contakes an open ball and have also closed set $\mathcal{H}_{T} = T(\mathcal{B}_{X}).$

Now T((Bx) is symmetric about 0 ad convex, and have so is By = T(Bx).

Observe $\not H_{\gamma} = \beta_{\Gamma}(\gamma, \gamma)$ and $H_{\gamma} = B_{r}(-r_{0},T)$ by symmetry about 0. But Then , I liwill v, - Yot w and yot w E Dy. By converdy w = $\frac{1}{2}(Y_0+w) + \frac{1}{2}(Y_0+w) \subset H_Y$ as used So $B_r(0,Y) \subseteq \mathcal{H}_Y = T(B_1(0,X))$. By the technical lemma, $B_{r_s}(0,Y) \subseteq T(B_r(0,X))$. But then for any $\varepsilon > 0$, $T(B_{\varepsilon}(0, X)) \ge B_{r_{\varepsilon}}(0, Y)$. Now let UEX be open and let yCT(U). Pick xellwith Tx=y. There exists ESO with BE(x,X)=U. $B_{u} + T_{h_{u}} + T(B_{\varepsilon}(u, X)) = T(u) + T(B_{\varepsilon}(o, X))$ $= T(x) + B_{rs}(0, Y)$ = Brig (y, y). So TU is open,