Baire Category Theorem

A complete metric space is not a countable union of nowhere dense sets.

Nowhere dense: $\overline{A}$ does not contain an open ball.

E.g. $\mathbb{Q}$ with usual metric.

Singletons are nowhere dense. $(a-\varepsilon, a+\varepsilon) \cap \mathbb{Q}$ is a ball.

$\mathbb{Q}$ is a countable union of its singletons, so it can't be

Let's apply, then return to it.

The inverse of a continuous linear map need not be continuous.

$$(\mathbb{Z}, \ell^1) \xrightarrow{id} (\mathbb{Z}, \ell^\infty)$$

$\|x\|_\infty \le \|x\|_1$ so is cts.

But $x_n = (\underbrace{1, \dots, 1}_{n}, 0, \dots)$

satisfies $\|x_n\|_\infty = 1$, $\|x_n\|_1 = n$ so id^{-1} is not cts.

Banach Isomorphism Theorem

Major result: If $T : X \to Y$ is continuous, linear, and bijective and if X, Y are Banach spaces, then T^{-1} is continuous.

HW: If X or Y is Banach and T, T^{-1} are cts, so is other.

The BIT follows from a more general, but more technical and less well motivated result:

Thm (Open Mapping Theorem)

Suppose $T \in B(X,Y)$, Y is a Banach space, and T is surjective. Then T is an open map.

(i.e. whenever $U \subseteq X$ is open in X

$T(U) = \{ T(x) : x \in U \}$ is open in Y.

Pf of BIT from OMT

The ball $B_1(0,X)$ is open so $T(B_1(0,X))$ is open and contains 0. So there is $r > 0$

$$\overline{B_r(0,Y)} \subseteq B_{2r}(0,Y) \subseteq T(B_1(0,X)).$$

Suppose $y \in Y$, $y \neq 0$.

Then $\frac{ry}{\|y\|} \in \overline{B_r}(0) \subseteq T(B_1(0,X))$

and $T^{-1}\left(\frac{ry}{\|y\|}\right) \in B_1(0,X)$.

I.e. $\left\| T^{-1}\left(\frac{ry}{\|y\|}\right)\right\|_X \leq 1$ and

$$\|T^{-1}y\|_X \leq \frac{1}{r}\|y\|_Y$$

Thus $\|T^{-1}\| \leq \frac{1}{r}$ and T^{-1} is continuous.

Main Technical Lemma   Hard work.

Prop: Suppose $T: X \to Y$, X is a Banach space and

$$\overline{T(B_1(0,X))} \supseteq B_r(0,Y)$$

then $T(B_1(0,X)) \supseteq B_{\frac{r}{2}}(0,Y)$.

Pf: Suppose $w \in B_{r/2}(0,Y)$, so $2w \in B_r(0,Y)$

Find $x_1 \in B_1(0,X)$, $\|2w - x_1\| < \frac{r}{2}$.

Since $2^2w - 2x_1 \in B_r(0,Y)$, we can find $x_2 \in B_1(0,X)$

$$\|2^2w - 2Tx_1 - Tx_2\| < \frac{r}{2}.$$

Continuing inductively, we can find $\{x_n\}$ $x_n \in B_1(0,X)$,

$$\|2^n w - 2^{n-1}Tx_1 - \cdots - 2^0 Tx_n\| < \frac{r}{2}.$$

Setting $z_n = \sum_{k=1}^{n} 2^{-k} x_k$ we find $\|w - Tz_n\| < \frac{r}{2^{n+1}}$.

Moreover the series $\sum_{k=1}^{\infty} 2^{-k} x_k$ is abs conv, so

converges to a limit z and $\|z\| \leq \sum 2^{-k}\|x_k\| < 1$.

Thus $z_n \to z \in B_1(0,X)$ and

$$Tz = \lim Tz_n = w.$$

Warmup exercises:

$A \subseteq X$ is symmetric about 0 if whenever $a \in A$, $-a \in A$.

Exercise: If A is symmetric about 0, so is $\overline{A}$.

Exercise: If $A \subseteq X$ is convex, so is $\overline{A}$.

Pf of OMT:

Suppose $T \in B(X,Y)$ is surjective and X and Y are Banach spaces. Then T is open.

Pf: Let $\mathcal{B}_X = B_1(0, X)$.

Let $\mathcal{G}_Y = \overline{T(\mathcal{B}_X)}$.

By linearity
$$r\mathcal{G}_Y = r\,\overline{T(\mathcal{B}_X)} = \overline{rT(\mathcal{B}_X)} = \overline{T(r\mathcal{B}_X)}.$$

Hence
$$\bigcup_{n\in\mathbb{N}} n\mathcal{G}_Y = \bigcup_{n\in\mathbb{N}} T(n\mathcal{B}_X) = T\Big(\bigcup n\mathcal{B}_X\Big) = T(X) = Y.$$

Since Y is complete, the Baire Category Theorem implies some $n\mathcal{G}_Y$ contains an open ball and hence also

closed set

$$\mathcal{G}_T = \overline{T(\mathcal{B}_X)}.$$

Now $T(B_X)$ is symmetric about 0 and convex, and hence so is $\mathcal{G}_Y = \overline{T(B_X)}$.

Observe $\mathcal{G}_Y \supseteq B_r(y_0, Y)$ and

$$\mathcal{G}_Y \supseteq B_r(-y_0, Y) \text{ by symmetry about } 0.$$

But then if $\|w\| < r$, $-y_0 + w$ and $y_0 + w \in \mathcal{G}_Y$.

By convexity $w = \frac{1}{2}(y_0 + w) + \frac{1}{2}(-y_0 + w) \in \mathcal{G}_Y$ as well

So $B_r(0, Y) \subseteq \mathcal{G}_Y = \overline{T(B_1(0, X))}$.

By the technical lemma, $B_{\frac{r}{2}}(0, Y) \subseteq T(B_1(0, X))$.

But then for any $\varepsilon > 0$, $T(B_\varepsilon(0, X)) \supseteq B_{\frac{r\varepsilon}{2}}(0, Y)$.

Now let $U \subseteq X$ be open and let $y \in T(U)$.

Pick $x \in U$ with $Tx = y$. There exists $\varepsilon > 0$ with $B_\varepsilon(x, X) \subseteq U$.

But then

$$\begin{aligned} T(B_\varepsilon(x, X)) &= T(x) + T(B_\varepsilon(0, X)) \\ &\supseteq T(x) + B_{\frac{r\varepsilon}{2}}(0, Y) \\ &= B_{\frac{r\varepsilon}{2}}(y, Y). \end{aligned}$$

So TU is open.