$L^1[a,b]$:

Start with $R[a,b]$. Riemann integrable functions.
Include continuous and piecewise continuous.

Fact: If $f \in R[a,b]$ and $\varepsilon > 0$ there is a continuous g, $\int_a^b |f-g| < \varepsilon$.

$(C[a,b], L^1)$ is fine, but not complete.

are Cauchy seq's that ought to converge to something discont.

$(R[a,b], L^1)$ is no good: not a norm!

$f = 0 \qquad g = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ on $[0,1]$

$\int |f| = 0, \quad \int |g| = 0.$ Oops.

$\hat{L}^1$ is equivalence classes of $R[a,b]$

$f \sim g$ if $\int |f-g| = 0$.

Integration is well defined on $\hat{L}^1$.

$$\int g = \int f + \int g - f$$

$$\left| \int g - f \right| \leq \int |g - f| = 0.$$

Exercise: $\hat{L}^1$ is a vectorspace and
$\|[f]\| \equiv \int |f|$ for any rep is
well defined and
a norm.

$$\frac{1}{\sqrt{x}} \geq M$$

$$\frac{1}{M^2} \geq x$$

Exercise $\hat{L}^1[a,b]$ is not complete.

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{x}} & x \geq \frac{1}{n} \\ 0 & 0 \leq x < \frac{1}{n} \end{cases}$$

. $[f_n]$ is Cauchy in $\hat{L}^1$ but does not
converge [if $|f| \leq M$. show $\|[f_n] - [f]\| \geq \int_0^{\frac{1}{M^2}} x^{-1/2} - \frac{1}{M}$
$= 2x^{1/2} \Big|_0^{1/M^2} = \frac{1}{M}$

$L^1[a,b]$ is the completion of $\hat{L}^1[a,b]$

For each $F \in L^1[a,b]$ is $\{f_n\}$ of Riemann integrable.

Cauchy in L^1 norm.

$$[f_n] \xrightarrow[L^1]{} F.$$

But F is just an abstract thing.

Still $\int_a^b F$ is well defined by extension

I will say $f : [a,b] \to \mathbb{R}$ is a representative of F

if there is a sequence in $R[a,b]$, $\{f_n\}$,

and $[f_n] \longrightarrow F$

$f_n \longrightarrow f$ pointwise.

More commonly, $\{f_n\}$ is Cauchy in L^1,

so it converges to some F, and $f_n \to f$ pointwise.

- If f is a rep of F we define
$$\int_a^b f = \int_a^b F = \lim \int_a^b f_n \quad \text{for any } f_n \to F.$$

Facts

$f_n \to 0$ p.w., f_n Cauchy $\Rightarrow$

* 1) Every $F \in L^1$ admits a representative f

($F = [f]$, $f \in \mathcal{R}[a,b]$ then f is a rep!)

* 2) There are (bounded!) functions $f: [a,b] \to \mathbb{R}$ that are not reps.

* 3) If $f \in \mathcal{R}[a,b]$ is a rep of F then $\int_a^b F = \int_a^b f$

[if $f_n \to f$ pointwise $\boxed{\text{and } \{f_n\} \text{ is Cauchy in } L^1}$,
$$\int_a^b f_n \to \int_a^b f \quad]$$

4) If f, g are reps of F, G $\quad f+g$ is a rep of $F+G$,
αf is a rep of αF

f_n, g_n conv F, G $\quad f_n \to f$ pw $\quad g_n \to g$ pw $\quad f_n + g_n \to F+G$

etc.

* 5) If f is a rep of $F \in L^1$ then

$|f|$ is a rep of an $H \in L^1$ and

$\int |f| = \|F\|_1$ We'll call $H = |F|$.

6) If f is a rep of F and f is a rep of G

then $G = F$ (If 0 is a rep of F, $F = 0$)

7 $[C[a,b]]$ is dense in $L^1[a,b]$

$[C_c(\mathbb{R})]$ is dense in $L^1(\mathbb{R})$ ← coming up ...

$\downarrow$

$= 0$ outside some interval

We will identify F with any of its reps.

Same game works for $1 \le p < \infty$.

Similar game for $\mathbb{R}$:

$\hat{L}^1$ equivalence classes of functions f,

$$f \in R[a,b] \quad \forall [a,b]$$

$$\int_{\mathbb{R}} |f| \equiv \lim \int_{-N}^{N} |f| \quad \text{finite}$$

$$f \sim g \quad \text{if} \quad \int_{\mathbb{R}} |f-g| = 0$$

Composition $S : X \to Y$
$T : Y \to Z$

$T \circ S : X \to Z$ is cts:

$$\|(T \circ S)(x)\|_Z \leq \|T\| \|Sx\|_Y$$
$$\leq \|T\| \|S\| \|x\|_X$$

So $\|T \circ S\| \leq \|T\| \|S\|$.

But strict inequality is possible:

$S(x,y) = (x,0)$ $\quad \|S\| = 1$
$T(x,y) = (0,y)$ $\quad \|T\| = 1$

$(T \circ S)(x,y) = (0,0)$ $\quad \|T \circ S\| = 0$.

$B(X) = B(X, X)$ If X is Banach so is $B(X)$!

Closed under addition but also composition.

Forms what is known as an algebra

(vector space with mult that gets along with vector space:

$$(A+B)C = AC + BC$$

$AB \neq BA$ in general, though.

Our mult is composition

This algebra has an identity, a map

I, $IA = AI = A$ $\forall$ A, namely the identity

Some elements of $B(X)$ admit inverses, and some do not.

$BA = I \leftarrow B$ is a left inverse

$AB = I \leftarrow B$ is a right inverse

both: B is an inverse.

We'll focus on continuous inverses and then see why it doesn't matter.

0 has none of the above.

$r(x) = (0, x_1, x_2, \ldots)$

$l(x) = (x_2, x_3, \ldots)$

$(l \circ r)(x) = x$

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l is r's left inverse

r is l's right inverse

l doesn't have a left inverse.

$(L \circ l)(x) = x$

$x = (1, 0, \ldots)$

$l(x) = 0$

$L(l(x)) = 0 \neq x$ no matter what x is

$(r \circ R)(1, 0, \ldots)$

$= (1, 0, \ldots)$

but $r(w) = (0, \ldots)$

Thm: If $A \in B(X)$ (X is Banach,) and $\|A\| < 1$ then

$I+A$ is invertible and

$$(I+A)^{-1} = \sum_{k=0}^{\infty} A^k$$

Pf: Note $\|A^k\| \leq \|A\|^k$. Since $\sum_{k=0}^{\infty} \|A\|^k$ converges to $\dfrac{1}{1-\|A\|}$

(this uses $\|A\|<1$), so does $\sum_{k=0}^{\infty} \|A^k\|$ and hence

the series is abs conv $\Rightarrow$ convergent.

Now $(I-A) \sum_{k=0}^{\cancel{\infty}\, N} A^k = I - A^{k+1}$

and $A^{N+1} \to 0$.

$AB_n \to AB \quad \|AB_n - AB\| \leq \|A\| \, \|B_n - B\|$!

So $(I-A) \sum_{k=0}^{\infty} A^k = I$.

But $\sum_{k=0}^{N} A^k (I-A) = I - A^{N+1}$ as well...

What is this saying?

I is the finest invertible operator around.

If we jiggle I by a bit

$$T = I - A$$

(A bit being $\|A\| < 1$) then the result is invertible as well.

Exercise: If B has a continuous inverse and $\|A\| \leq \|B^{-1}\|^{-1}$ then $B + A$ has a continuous inverse

$$B + A = B\left[I + B^{-1}A\right]$$

$$\|B^{-1}A\| \leq \|B^{-1}\| \|A\|$$

Cor: The set of elements of $B(X,Y)$ with continuous inverses is open.

Pf: Suppose $T \in B(X,Y)$ has an inverse T^{-1}.

Suppose $\|A\| < \|T^{-1}\|^{-1}$.

Observe $T + A = T(I + T^{-1}A)$.

Moreover, $\|T^{-1}A\| \leq \|T^{-1}\| \|A\| < 1$.

So $T+A$ is a composition of continuous invertible maps and is invertible: (and cts).

$(T+A)^{-1} = (I + T^{-1}A)^{-1} T^{-1}$.

But then there is a ball of radius $\varepsilon = \|T^{-1}\|^{-1}$ about T inside of the set of cts invertible maps.

Baire Category Theorem

A complete metric space is not a countable union of nowhere dense sets.

Nowhere dense: $\overline{A}$ does not contain an open ball.

E.g. $\mathbb{Q}$ with usual metric.

Singletons are nowhere dense. $(a-\varepsilon, a+\varepsilon) \cap \mathbb{Q}$ is a ball.

$\mathbb{Q}$ is a countable union of its singletons, so it can't be

Let's apply, then return to it.