Thm: If Y is Banach,

$B(X,Y)$, with operator norm, is a Banach space.

Pf: Let $\{T_n\}$ be Cauchy in $B(X,Y)$.

Let $x \in X$.

$$\|T_n x - T_m x\| = \|(T_n - T_m)(x)\|$$
$$\leq \|T_n - T_m\| \, \|x\|$$

So $\{T_n x\}$ is Cauchy in Y and converges to a limit Tx.

I claim that T is linear, bounded, and $T_n \to T$.

For $x, z \in X$, $T(x+z) = \lim T_n(x+z)$
$$= \lim T_n x + T_n z$$
$$= Tx + Tz.$$

etc.

Recall Cauchy sequences are bounded, so $\exists\, M$, $\|T_n\| \leq M$

Notice $\quad \|Tx\| = \lim \|T_n x\|$

Since $\|T_n x\| \leq \|T_n\|\,\|x\|$

$\leq M\|x\|$ for all n,

$$\|Tx\| \leq M\,\|x\|$$

So $\|T\| \leq M$ and T is bounded.

To show $T_n \to T$ suppose $\|x\| \leq 1$. Observe

$$\|(T_n - T)(x)\| = \lim_{m\to\infty} \|(T_n - T_m)x\|.$$

Let $\varepsilon > 0$.
Pick N so if $n, m \geq N$ $\quad \|T_n - T_m\| < \varepsilon$

So $\quad \|(T_n - T)(x)\| \leq \varepsilon\,\|x\| \leq \varepsilon$ if $n \geq N$.

I.e. $\quad \|T_n - T\| \leq \varepsilon$ if $n \geq N$.

Prop: Suppose X is a normed linear space. Then there exists a Banach space Y and a map $T: X \to Y$ such that

1) $\|Tx\|_X = \|x\|_Y \quad \forall x \in X$

2) $\overline{TX} = Y$

e.g. $\mathbb{Z}$ with ℓ^2 norm, has ℓ^2 as a completion.

We often visualize $X \subseteq Y$ by id. with TX

① Such a map is called an isometry. It preserves distance.

Pf: Let $\tilde{Y}$ be the set of Cauchy sequences in Y.

I will identify two cauchy sequences $(x_1, \dots)$ $(z_1, \dots)$

if $(x_1, z_1, \dots)$ is still Cauchy, and write $(x_n) \sim (z_n)$

Boring: $(x_n) \sim (x_n)$

$(x_n) \sim (y_n) \Rightarrow (y_n) \sim (x_n)$

$(x_n) \sim (y_n),\ (y_n) \sim (z_n) \Rightarrow (x_n) \sim (z_n)$

Y: set of equiv classes.

Exercise: Y is a vector space

Note $\big| \|x_n\| - \|x_m\| \big| \leq \|x_n - x_m\|$

So $\lim \|x_n\|$. exists.

Exercise: if $(x_n) \sim (z_n)$, $\lim \|(x_n)\| = \lim \|(z_n)\|$.

$\|[(x_n)]\|$ is $\lim \|x_n\|$.

Exercise: $\|\cdot\|$ is a norm on Y.

e.g. $\|[(x_n)]\| = 0 \Rightarrow \lim \|x_n\| = 0$

$\Rightarrow \lim x_n = 0$

$\Rightarrow (x_1, 0, x_2, 0, \ldots) \to 0$

$\Rightarrow (x_n) \sim (0)$.

Map from X to Y: $x \to (x)$. Clearly an isometry

Image of X is dense:

$(x_1, \dots, \quad)$ $\qquad \epsilon > 0$

x_N

$x_n - x_N$

$\uparrow$

$\le \epsilon \;\; \forall n$ large.

Y is complete.

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We call Y a completion of X.

If Y_1 and Y_2 are completions

$T_1 : X \to Y_1$

$T_2 : X \to Y_2$

$T_2 \circ T_1^{-1} : T_1 X \to Y_2$

is an isometry onto its image.

image is dense

Thm: Suppose X is a normed space, Y is a Banach space, and suppose W is dense in X.

Given $S \in B(W, Y)$ $\exists!\, T \in B(X, Y)$ $T|_W = S$, and $\|T\| = \|S\|$

Application: Completions are essentially unique.

$$X \xrightarrow{T_1} Y_1$$

$$X \xrightarrow{T_2} Y_2$$

isometries

$$\overline{T_i X} = Y_i$$

$$T_2 \circ T_1^{-1} : T_1(X) \longrightarrow Y_2$$

$$\uparrow$$

$$\overline{T_1(X)} = Y_1$$

is $T : Y_1 \to Y_2$, unique

$$T(T_1(x)) = T_2(x)$$

$\|y_1\| = \lim \|x_n\|$

Moreover: $T_1(x_n) \to y_1$

$$\|T y_1\| = \lim \|T T_1 x_n\|$$
$$= \lim \|T_2 x_n\| = \lim \|x_n\|$$

Pf: Let $x \in X$ and pick $w_n \to x$.

Observe $\big| \|Sw_n - Sw_m\| \big| \leq \|S(w_n - w_m)\|$

$$\leq \|S\| \, \|w_n - w_m\|$$

So $\|Sw_n\|$ is cauchy and converges to a limit y

Moreover, if $\hat{w}_n \to x$ then

$$w_n - \hat{w}_n \to 0$$

$$S(w_n - \hat{w}_n) \to 0 \quad \text{by continuity.}$$

$$Sw_n \to y \qquad S\hat{w}_n \to \hat{y}$$

$$S(w_n - \hat{w}_n) \to y - \hat{y} \implies y = \hat{y}.$$

We define $Tx = \lim Sw_n$.

Need to verify a) T is linear

b) T is bounded.

c) $\|T\| = \|S\|$

a: Given x, $\hat{x}$

$$w_n \to x \quad \hat{w}_n \to \hat{x}$$

$$T(x+\hat{x}) = \lim S(w_n + \hat{w}_n)$$

$$= \lim S w_n + \lim S \hat{w}_n$$

$$= Tx + T\hat{x}.$$

ditto for scalar mult.

b) $$\|Tx\| = \lim \|S w_n\|$$

But $\|S w_n\| \leq \|S\| \, \|w_n\| \longrightarrow \|S\| \, \|x\|$.

So $\|Tx\| \leq \|S\| \, \|x\|$. So $\|T\| \leq \|S\|$.

c) Exercise: $\|T\| \geq \|S\|$.

Finally, if T' is another such given $x \in X$,

$$T'(x) = \lim T'(w_n) = \lim S(w_n) = Tx.$$

So T is unique.