This is just the identity map.

Lemma: Suppose $T: X \to Y$ is continuous at $x = 0$.
Then T is continuous.

Pf: Suppose $x_n \to x$ in X.

Then $x_n - x \to 0$ in X and

$$T(x_n - x) \to T(0) = 0 \text{ in } Y.$$

But $T(x_n - x) = T(x_n) - T(x)$ for all n.

Thus $\lim T(x_n) - T(x) = 0$ and

$$\lim T(x_n) = T(x).$$

Upshot: Not continuous at $0 \Rightarrow$ not continuous.

Of course, if cts $\Rightarrow$ cts at 0.

So T is continuous iff it is cts at 0.

Lemma: If $T: X \to Y$ is continuous, then there exists $K > 0$ such that

$$\| T(x) \|_Y \leq K \|x\|_X \quad \text{for all } x \in X$$

Pf: By continuity, there exists $\delta > 0$ so if $\|x - 0\|_X < \delta$,

$$\| T(x) - T(0) \| < 1.$$

I.e. if $\quad \|x\|_X < \delta \Rightarrow \|T(x)\|_Y < 1$

Let $K = \frac{2}{\delta}$ and suppose $x \neq 0$.

Then $z = \dfrac{x}{K\|x\|}$ satisfies $\|z\| = \dfrac{\delta}{2} < \delta$.

So $\| T z \|_Y < 1$.

Hence $\left\| T \dfrac{x}{K\|x\|} \right\| < 1$ and $\|T(x)\| < K\|x\|$.

This still holds for $x = 0$ also and we are done.

Cor: If T is cts, there is a K,

$$\|T(x)\| \leq K$$

for all $x \in X$, $\|x\| \leq 1$.

The ball of radius 1 is sent inside the ball of radius K.

Lemma: Suppose $\exists\, K$, $\|Tx\| \leq K$ for all $x \in X$, $\|x\| \leq 1$. Then $\|Tx\| \leq K\|x\|$ $\forall x \in X$.

Pf: Suppose $x \neq 0$. Then $\left\| x/_{\|x\|} \right\| = 1$ and

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq K \Rightarrow \|T(x)\| \leq K\|x\|.$$

Def: We say $T: X \to Y$, linear, is bounded if $\exists\, K$, $\|Tx\| \leq K$ $\forall x$, $\|x\| \leq 1$.

or $\|Tx\| \leq K\|x\|$ $\forall x$.

We've proved:
If T is cts $\Rightarrow$ T is bounded

Prop: If T is bounded, then T is cts.

Pf: Suppose $x_n \to 0$.

Then $0 \leq \|Tx_n\| \leq K\|x_n\| \to 0$.

So T is cts at 0, and hence cts.

Let's redo that example:

$$(Z, \ell^{\infty}) \to (Z, \ell^{1})$$

$$x_n = (\underbrace{1, \ldots, 1}_{n}, 0, \ldots)$$

$\|x_n\|_{\infty} = 1$

$\|Tx_n\|_1 = n$

$\Rightarrow$ not bounded! $\Rightarrow$ not cts.

So for no K $\|Tx\|_1 \leq K\|x\|_{\infty}$, $\forall x \in Z$

In summary:

Thm: Given a linear map $T: X \to Y$, TFAE

a) T is continuous

b) T is continuous at 0

c) $\exists K > 0$ $\|Tx\| \leq K$ $\forall x$, $\|x\| \leq 1$

d) $\exists K > 0$, $\|Tx\| \leq K\|x\|$ for all $x \in X$.

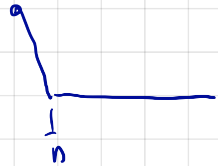
$C[0,1], L^\infty$ vs $C[0,1], L^1$

A   B

$A \to B$ is cts

$B \to A$ is not cts

e.g. f_n :



$\|f_n\|_1 = \frac{1}{2n} \to 0$. So $f_n \xrightarrow{L_1} 0$.

But $\|f_n\|_\infty = 1$ for all n. So $f_n \not\xrightarrow{L_\infty} 0$.

($1 \le K \frac{1}{2n}$ $\forall n$ is impossible)

Suppose $\|f\|_\infty \le 1$.

Then $\|f\|_1 = \int_0^1 |f| \le \int_0^1 1 = 1$.

Thus $B \to A$ is bounded + hence cts.

E.g. $\ell^p \hookrightarrow \ell^\infty$ is cts.

($x \in \ell^p \Rightarrow x(k) \to 0$, so $x \in \ell^\infty$.

The question is continuity).

e.g. $\quad y \in \ell^2$

$$T(x) = \langle x, y \rangle. \qquad T : \ell^2 \to \mathbb{R}$$

$$|T(x)| \leq \underbrace{\|y\|_2}_{K} \|x\|_2 \qquad \text{By C-S.}$$

So T is cts.

e.g. $\quad y \in \ell^\infty$

$$T : \ell^1 \to \mathbb{R}$$

$$T(x) = \sum y_k x_k$$

$$|T(x)| \leq \sum |y_k| |x_k| \leq \|y\|_\infty \sum |x_k|$$

$$= \underbrace{\|y\|_\infty}_{K} \|x\|_1$$

e.g: $\quad y \in \ell^\infty \qquad T : \ell^1 \to \ell^1$

$$Tx = (y_1 x_1, y_2 x_2, \ldots).$$

$$\|Tx\|_1 = \sum |x_k y_k| \leq \|y\|_\infty \|x\|_1 \quad \text{as above.}$$

Much harder:

$\ell^2 \subseteq \ell^\infty$ $\qquad \sum |x_k|^2 < \infty$

$\Rightarrow \; x_k \to 0.$

Is this map continuous?

(Stay Tuned for the Banach Iso Thm).

$\mathcal{P} \to$ polynomials on $[0,1]$ $\qquad L^\infty$ norm.

$I(p) = \int_0^x p(t)\, dt. \qquad I : \mathcal{P} \to \mathcal{P}.$

$$\left| \left(I(p) \right)(x) \right| = \left| \int_0^x p(t) \right| \leq |x| \, \|p\|_\infty \leq \|p\|_\infty$$

I.e. $\|I(p)\|_\infty \leq \|p\|_\infty$, so I is cts.

Derivatives?

$$p_n(x) = x^n \qquad \|p_n\|_\infty = 1$$

$$\|D(p_n)\| = n\,\|x^{n-1}\|$$

$$= n$$

So image of unit ball is unbounded.

So not continuous (!).

Control in L^∞ is silent about derivatives!

Notation: $B(X,Y)$: continuous (bounded) linear maps from X to Y.

This is a vector space in a natural way!

If $\|Tx\| \leq K\|x\| \quad \forall x$ and $K' \geq K$, $\|Tx\| \leq K'\|x\|$ also.

But the least such K might be intesting.

$$k = \inf\{K : \|Tx\| \leq K\|x\| \;\forall x \in X\},$$

For any x, if $K \geq k$ $\quad \|Tx\| \leq \cancel{K}\|x\| \quad \forall x \in X, \neq 0$

$\|Tx\| \leq k\|x\|$ also.

$$\frac{\|Tx\|}{\|x\|} \leq k \quad \text{for all } x \neq 0.$$

And in fact $k = \sup\limits_{x \neq 0} \dfrac{\|Tx\|}{\|x\|}$.

Def: Given a linear $T: X \to Y$,

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

I claim $\|\cdot\|$ is a norm on $B(X, Y)$.

Evidently $\|T\| \geq 0$.

If $\|T\| = 0 \Rightarrow \|Tx\| = 0 \ \forall x \neq 0$ and $T = 0$.

$$\|\alpha T\| = \sup_{x \neq 0} \frac{\|\alpha T x\|}{\|x\|} = \sup_{x \neq 0} \frac{|\alpha| \, \|Tx\|}{\|x\|}$$

$$= |\alpha| \, \|T\|.$$

As for Δ,

$$\|(T+S)(x)\| \leq \|Tx\| + \|Sx\|$$

$$\leq \|T\| \, \|x\| + \|S\| \, \|x\|.$$

S_

for $x \neq 0$

$$\frac{\|(T+S)x\|}{\|x\|} \leq \|T\| + \|S\|$$

So $\sup \dfrac{\|(T+S)x\|}{\|x\|} \leq \|T\| + \|S\|.$

Note: if $x \neq 0$ and $z = \frac{x}{\|x\|}$,

$$\frac{\|Tx\|}{\|x\|} = \frac{\|Tz\|}{\|z\|}$$

So $\displaystyle\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$

Also, $\displaystyle\sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$

$$\|T\| = \begin{cases} \displaystyle\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \\ \displaystyle\sup_{\|x\|=1} \|Tx\| \\ \displaystyle\sup_{\|x\| \leq 1} \|Tx\| \end{cases}$$

How big can the unit ball be stretched?