Cor: If $x \in X$ and $c_k = \langle x, e_k \rangle$,

then $\sum_{k=1}^{\infty} c_k e_k$ converges.

(Previous two results).

So, then, when can we guarantee

$$x = \underbrace{\sum_{k=1}^{\infty} \langle x_k, e_k \rangle e_k}_{y} \; ?$$

$z = x - y$. $\quad \langle z, e_k \rangle = \langle x - y, e_k \rangle$

$$= \langle x, e_k \rangle - \langle y, e_k \rangle$$

$$\langle y, e_k \rangle = \lim \langle s_n, e_k \rangle = \langle x, e_k \rangle.$$

So $\quad z \in \{e_1, \dots\}^{\perp}$

$z \in Sp(e_1, \dots)^{\perp}$

$z \in \overline{Sp}(e_1, \dots)^{\perp}$

If $w \in \overline{Sp}(e_1, \dots)$ find $w_k \in Sp(e_1, \dots)$

$w_k \to w$.

$z \perp w_k$ for all k

$\langle z, w \rangle = \lim \langle z, w_k \rangle = 0$.

We say an o.n. sequence is *complete* if

$$\overline{Sp}\{e_1, \dots\}^{\perp} = \{0\}.$$

Prop: If $e_1, \dots$ is a complete o.n. sequence then if $x \in X$,

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$$

Pf: Let $y =$. By our observations above

$x - y \in \overline{Sp}\{e_1, \dots\}^{\perp} = \{0\}$. So $x = y$.

Prop: Suppose e_k is an o.n. sequence and

$x = \sum \langle x, e_k \rangle e_k \quad \forall \; x \in X.$

Then the sequence is complete.

Pf: Suppose the sequence is not complete.

Pick $z \neq 0$, $z \in \overline{Sp}\{e_1, \dots, e_n\}^{\perp}$.

Observe $\langle z, e_k \rangle = 0 \quad \forall k.$

But $z \neq \sum \langle z, e_k \rangle e_k$.

Lemma: TFAE

1) $\{e_n\}$ is complete
2) $\{e_n\}^{\perp} = \{0\}$
3) $\overline{Sp}(e_n) = X$.

Pf: 1) $\Rightarrow$ 2) Suppose $\{e_n\}$ is complete.
Suppose $\langle x, e_k \rangle = 0 \quad \forall k$.
Then $x = \sum \langle x, e_k \rangle e_k = 0$. So $\{e_n\}^{\perp} = \{0\}$.

2) $\Rightarrow$ 1) $\{e_n\} \subseteq \overline{Sp}\{e_n\} \Rightarrow \{0\} = \{e_n\}^{\perp} \supseteq \overline{Sp}\{e_n\}^{\perp}$.

1) ⇒ 3) $\overline{Sp}\{e_n\}^\perp = \{0\}$

$$\Rightarrow \left(\overline{Sp}\{e_n\}^\perp\right)^\perp = \{0\}^\perp$$

$$\Rightarrow \overline{Sp}\{e_n\} = X.$$

closed subspace!

3) ⇒ 1) $\overline{Sp}\{e_n\} = X \Rightarrow \overline{Sp}\{e_n\}^\perp = X^\perp = \{0\}.$

Prop: $\{e_k\}$ is complete iff for all $x \in X$,

$$\|x\|^2 = \sum |\langle x, e_k\rangle|^2. \leftarrow \text{Parsevals Identity}$$

Pf: If not complete, $\exists\ x$, $x \neq 0$, $\langle x, e_k\rangle = 0\ \forall k$.

So $\|x\|^2 \neq \sum |\langle x, e_k\rangle|^2$.

Suppose $\{e_k\}$ is complete so

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, e_k\rangle e_k.$$

Then $\|x\|^2 = \| \lim \text{—} \|^2$

$$= \lim_{n\to\infty} \left\| \sum_{k=1}^{n} \langle x, e_k\rangle e_k \right\|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} |\langle x, e_k\rangle|^2$$

$$= \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2.$$

Bessel: $\sum |\langle x, e_k \rangle|^2 \leq \|x\|^2$ generally

Parseval: $\sum |\langle x, e_k \rangle|^2 = \|x\|^2$ for a <u>complete</u> o.n. sequence.

Def: An o.n. basis for a Hilbert space is a complete o.n. sequence.

Prop: A infinite-dim Hilbert space is separable iff it admits an o.n. basis.

Sketch: separable: $\{x_n\}$ dense.

reduce to a lin ind set. ; the span is still dense

Perform Gram-Schmidt. The span is the same, and hence dense. $\Rightarrow$ complete

If $\{e_k\}$ is an o.n. basis, I claim

$$\{x : \underset{\exists n,}{\uparrow} \sum_{k=1}^{n} q_k e_k \quad q_k \in \mathbb{Q}\} \text{ is dense.}$$

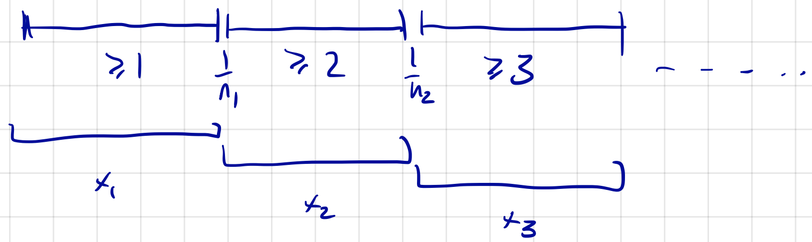
Continuity:

There are linear maps $T: X \to Y$ that are not continuous.
($x_n \to x \Rightarrow T(x_n) \to T(x)$).

Z with the ℓ^∞ norm.

Z with the ℓ^1 norm.

$(1, \frac{1}{2}, \frac{1}{3}, \dots \quad)$



$\|x_k\|_\infty = \frac{1}{n_k} \le \frac{1}{k} \to 0 \qquad x_k \to 0 \text{ . } (Z, \ell^\infty)$

But $\|x_k\|_1 \to \infty$ so $x_k \to x$ for any x in ℓ^1!

This is just the identity map.

Lemma: Suppose $T: X \to Y$ is continuous at $x = 0$.
Then T is continuous.

Pf: Suppose $x_n \to x$ in X.

Then $x_n - x \to 0$ in X and

$$T(x_n - x) \to T(0) = 0 \text{ in } Y.$$

But $T(x_n - x) = T(x_n) - T(x)$ for all n.

Thus $\lim T(x_n) - T(x) = 0$ and

$$\lim T(x_n) = T(x).$$

Upshot: Not continuous at $0 \Rightarrow$ not continuous.

Of course, if cts $\Rightarrow$ cts at 0.

So T is continuous iff it is cts at 0.

Lemma: If $T: X \to Y$ is continuous, then there exists $K > 0$ such that

$$\|T(x)\|_Y \leq K \|x\|_X \quad \text{for all } x \in X$$

Pf: For $\varepsilon = 1$, there exists $\delta > 0$ so if $\|x - 0\|_X < \delta$,

$$\|T(x) - T(0)\| < 1.$$

I.e. if $\|x\|_X < \delta \Rightarrow \|T(x)\|_Y < 1$

Let $K = \frac{2}{\delta}$ and suppose $x \neq 0$.

Then $z = \frac{x}{K\|x\|}$ satisfies $\|z\| = \frac{\delta}{2} < \delta$.

So $\|Tz\|_Y < 1$.

Hence $\left\| T \frac{x}{K\|x\|} \right\| < 1$ and $\|T(x)\| < K\|x\|$.

This still holds for $x = 0$ also and we are done.

Cor: If T is cts, there is a K,

$$\|T(x)\| \leq K$$

for all $x \in X$, $\|x\| \leq 1$.

The ball of radius 1 is sent inside the ball of radius K.

Lemma: Suppose $\exists K$, $\|Tx\| \leq K$ for all $x \in X$, $\|x\| \leq 1$. Then $\|Tx\| \leq K\|x\|$ $\forall x \in X$.

Pf: Suppose $x \neq 0$. Then $\|x/\|x\|\| = 1$ and

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq K \Rightarrow \|T(x)\| \leq K\|x\|.$$

Def: We say $T: X \to Y$, linear, is bounded if $\exists K$, $\|Tx\| \leq K$ $\forall x$, $\|x\| \leq 1$.

If T is cts $\Rightarrow$ T is bounded

Prop: If T is bounded, then T is cts.

Pf: Suppose $x_n \to 0$.

Then $0 \leq \|Tx_n\| \leq K\|x_n\| \longrightarrow 0$.

So T is cts at 0, and hence cts.

Let's redo that example:

$$(\mathbb{Z}, \ell^\infty) \to (\mathbb{Z}, \ell^1)$$

$$x_n = (\underbrace{1, \dots, 1}_{n}, 0, \dots)$$

$\|x_n\|_\infty = 1$

$\|Tx_n\|_1 = n$

$\Rightarrow$ not bounded! $\Rightarrow$ not cts.

In summary:

Thm: Given a linear map $T: X \to Y$, TFAE

a) T is continuous

b) T is continuous at 0

c) T is bounded

d) $\exists\, K > 0$, $\|Tx\| \le K\|x\|$ for all $x \in X$.

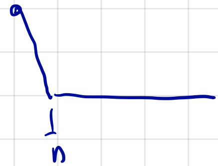
$C[0,1], L^\infty$ vs $C[0,1], L^1$

A B

$A \to B$ is cts

$B \to A$ is not cts

e.g. f_n :



$\|f_n\|_1 = \frac{1}{2n} \to 0$. So $f_n \xrightarrow{L_1} 0$.

But $\|f_n\|_\infty = 1$ for all n. So $f_n \overset{L_\infty}{\not\to} 0$.

Suppose $\|f\|_\infty \leq 1$.

Then $\|f\|_1 = \int_0^1 |f| \leq \int_0^1 1 = 1$.

Thus $B \to A$ is bounded + hence cts.

E.g. $\ell^p \hookrightarrow \ell^\infty$ is cts.

($x \in \ell^p \Rightarrow x(k) \to 0$, so $x \in \ell^\infty$.

The question is continuity).

e.g. $y \in \ell^2$

$T(x) = \langle x, y \rangle$. $\quad T: \ell^2 \to \mathbb{R}$

$|T(x)| \leq \underbrace{\|y\|_2}_{K} \|x\|_2$ By C-S.

So T is cts.

e.g. $y \in \ell^\infty$

$T: \ell^1 \to \mathbb{R}$

$T(x) = \sum y_k x_k$

$$|T(x)| \leq \sum |y_k| |x_k| \leq \|y\|_\infty \sum |x_k|$$

$$= \underbrace{\|y\|_\infty}_{K} \|x\|_1$$

e.g: $y \in \ell^\infty$ $\quad T: \ell^1 \to \ell^1$

$Tx = (y_1 x_1, y_2 x_2, \dots)$.

$\|Tx\|_1 = \sum |x_k y_k| \leq \|y\|_\infty \|x\|_1$ as above.

Much harder:

$$\ell^2 \subseteq \ell^\infty \qquad \sum |x_k|^2 < \infty$$

$$\Rightarrow \quad x_k \to 0.$$

Is this map continuous?

(Stay Tuned for the Banach Iso Thm).

$\mathcal{P} \to$ polynomials on $[0,1]$ $\quad L^\infty$ norm.

$$I(p) = \int_0^x p(t)\, dt. \qquad I : \mathcal{P} \to \mathcal{P}.$$

$$\left| \left(I(p) \right)(x) \right| = \left| \int_0^x p(t) \right| \leq |x| \, \|p\|_\infty \leq \|p\|_\infty$$

I.e. $\quad \| I(p) \|_\infty \leq \|p\|_\infty.$

Derivatives?

$$p_n(x) = x^n \qquad \|p_n\|_\infty = 1$$

$$\|D(p_n)\| = n \; \|x^{n-1}\|$$

$$= n$$

So image of unit ball is unbounded.