Def: A collection $e_1, e_2, \ldots$ of vectors is called an o.n. system if $\langle e_i, e_j \rangle = \delta_{ij}$

Given $x \in X$,

consider $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$. Does this converge?

First step:

Bessel's inequality

$$\langle a e_1 + b e_2, e_1 \rangle = a$$

$$\langle e_1, a e_1 + b e_2 \rangle = \bar{a}$$

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

Pf: Let $x_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$. Observe $\|x_n\|^2 = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$.

$$\|x - x_n\|^2 = \langle x - x_n, x - x_n \rangle$$

$$= \|x\|^2 - \langle x_n, x \rangle - \langle x, x_n \rangle + \|x_n\|^2.$$

But: $\langle x_n, x \rangle = \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, x \rangle = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 = \|x_n\|^2$.

Also $\langle x, x_n \rangle = \overline{\langle x_n, x \rangle} = \overline{\|x_n\|^2} = \|x_n\|^2$.

So $\|x - x_n\|^2 = \|x\|^2 - \|x_n\|^2$ and $\|x_n\|^2 = \|x\|^2 - \|x - x_n\|^2 \leq \|x\|^2$.

Now: $y_n = \sum_{k=1}^{n} \langle x, e_n \rangle e_n$

$$\|y_n - y_m\|^2 = \underbrace{\sum_{k=n+1}^{m} |\langle x, e_k \rangle|^2}$$

This is small for n, m large by Cauchy criterion for series.

So $\{y_n\}$ is Cauchy and converges to a limit.

So who can we represent this way?

Certainly $\operatorname{Span}(e_1, \ldots)$.

Moreover, every x with such an expansion is in $\overline{\operatorname{Span}(e_1, \ldots)}$.

$x_n \longrightarrow x$

eg: ℓ^2

$e_1 = (1, 0, 0, \ldots)$
$e_2 = (0, 1, 0, \ldots$
etc.

$\langle e_i, e_j \rangle = \delta_{ij}$

$\mathrm{Span}(e_1, e_2, \ldots) = Z$

The elements e_i do not form a basis for ℓ^2.

But they form something better:

$x = (x_1, x_2, \ldots)$

$\sum |x_i|^2 < \infty.$

$x = \sum_{i=1}^{\infty} x_i e_i$ How?

$X^k = \sum_{i=1}^{k} x_i e_i$ Cauchy: $k \le \ell$ $X^k - X^\ell = \sum_{i=k+1}^{\ell} x_i e_i$

$\|X^k - X^\ell\|^2 = \sum_{i=k+1}^{\ell} |x_i|^2$

$\hookrightarrow$ small by Cauchy crit.

$$x - x^k = (0, 0, \dots, 0, x_{k+1}, \dots)$$

$$\|x - x^k\|^2 = \sum_{i=k+1}^{\infty} |x_i|^2 \longrightarrow 0$$

$$\hookrightarrow \left| \sum_{i=1}^{\infty} |x_i|^2 - \sum_{i=1}^{k} |x_i|^2 \right| \longrightarrow 0$$

And the coefficients $x_k = \langle x, e_k \rangle$.

And $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$.

We'll now show that we can do this kind of thing more generally in a Hilbert space.

Given a Hilbert space, an o.n. sequence in X is a sequence $e_1, e_2, \ldots$ of vectors $\langle e_i, e_j \rangle = 0$

Prop: Every infinite dimensional inner prod space admits an o.n. sequence.

Pf: Start with any $x_1 \neq 0$.

$$f_1 = x_1 \qquad e_1 = f_1 / \|f_1\|.$$

Since X is inf dim, $\exists\, x_2$, x_2 not in span of e_1.

$$f_2 = x_2 - \langle x_2, e_1 \rangle e_1 \qquad e_2 = x_2 / \|x_2\|.$$

Since X is inf dim, ...

Exercise:

In $C[0,\pi]$ with L^1 norm,

$\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}} \cos(kx)$ form an o.n. sequence.

Given an o.n. sequence, and $x \in X$,

we'd like to write $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$.

There's no reason to expect this: $\{e_2, e_3, \ldots\} \subseteq \ell^2$ is an o.n. sequence but $e_1 \neq \sum_{k=2}^{\infty} \langle e_1, e_k \rangle e_k = 0$.

You might not have enough e's! We'll first start by showing that at least $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ adds up to something.

Lemma (Bessel's Inequality): Let $x \in X$, an IP space, and let $e_1, \ldots$ be an o.n. sequence in X. Then

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

Pf: Let $x_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$. So $\|x_n\|^2 = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$.

Observe $\|x - x_n\|^2 = \|x\|^2 - 2 \operatorname{Re} \langle x, x_n \rangle + \|x_n\|^2$.

Moreover, $\langle x, x_n \rangle = \sum_{k=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_k \rangle} = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$

$= \|x_n\|^2$.

Thus $\quad \|x_n\|^2 + \|x - x_n\|^2 = \|x\|^2$ and

$$\|x_n\|^2 \le \|x\|^2.$$

Lemma: Suppose $(c_1, c_2, \dots) \in \ell^2$, and $\{e_1, \dots\}$ is an on sequence in a Hilbert space X. Then

$$\sum_{k=1}^{\infty} c_k e_k \quad \text{conveges.}$$

Pf: Let $\displaystyle s_n = \sum_{k=1}^{n} c_k e_k.$

Then, if $m > n$, $\quad \displaystyle s_m - s_n = \sum_{k=n+1}^{m} c_k e_k$

and $\quad \displaystyle \|s_m - s_n\|^2 = \sum_{k=n+1}^{m} |c_k|^2.$

But by the Cauchy criterion for series we conclud

$\{s_n\}$ is Cauchy. Since X is complete, $s_n \to x$ for some $x \in X$.

Cor: If $x \in X$ and $c_k = \langle x, e_k \rangle$,

then $\sum_{k=1}^{\infty} c_k e_k$ converges.

(Previous two results).

So, then, when can we guarantee

$$x = \underbrace{\sum_{k=1}^{\infty} \langle x_k, e_k \rangle e_k}_{y} \; ?$$

$z = x - y$. $\quad \langle z, e_k \rangle = \langle x - y, e_k \rangle$

$$= \langle x, e_k \rangle - \langle y, e_k \rangle$$

$$\langle y, e_k \rangle = \lim \langle s_n, e_k \rangle = \langle x, e_k \rangle.$$

So $\quad z \in \{e_1, \dots\}^{\perp}$

$z \in Sp(e_1, \dots)^{\perp}$

$z \in \overline{Sp}(e_1, \dots)^{\perp}$

If $w \in \overline{Sp}(e_1,\dots)$ find $w_k \in Sp(e_1,\dots)$

$w_k \to w$.

$z \perp w_k$ for all k

$\langle z,w\rangle = \lim \langle z,w_k\rangle = 0$.

We say an o.n. sequence is <u>complete</u> if

$$\overline{Sp}\{e_1,\dots\}^\perp = \{0\}.$$

Prop: If $e_1,\dots$ is a complete o.n. sequence then if $x\in X$,

$$x = \sum_{k=1}^{\infty} \langle x,e_k\rangle e_k$$

Pf: Let $y =$. By our observations above

$x-y \in \overline{Sp}\{e_1,\dots\}^\perp = \{0\}$. So $x=y$.

Prop: Suppose e_k is an o.n. sequence and

$$x = \sum \langle x, e_k \rangle e_k \quad \forall \; x \in X.$$

Then the sequence is complete.

Pf: Suppose the sequence is not complete.

Pick $z \neq 0$, $z \in \overline{Sp}\{e_1, \ldots, e_n\}^{\perp}$.

Observe $\langle z, e_k \rangle = 0 \quad \forall k$.

But $z \neq \sum \langle z, e_k \rangle e_k$.

Lemma: TFAE

1) $\{e_n\}$ is complete
2) $\{e_n\}^{\perp} = \{0\}$
3) $\overline{Sp}(e_n) = X$.

Pf: 1) $\Rightarrow$ 2) Suppose $\{e_n\}$ is complete.
Suppose $\langle x, e_k \rangle = 0 \quad \forall k$.
Then $x = \sum \langle x, e_k \rangle e_k = 0$. So $\{e_n\}^{\perp} = \{0\}$.

2) $\Rightarrow$ 1) $\{e_n\} \subseteq \overline{Sp}\{e_n\} \Rightarrow \{0\} = \{e_n\}^{\perp} \supseteq \overline{Sp}\{e_n\}^{\perp}$.

1) ⇒ 3) $\overline{Sp}\{e_n\}^{\perp} = \{0\}$

$$\Rightarrow \left(\overline{Sp}\{e_n\}^{\perp}\right)^{\perp} = \{0\}^{\perp}$$

$$\Rightarrow \overline{Sp}\{e_n\} = X.$$

closed subspace!

3) ⇒ 1) $\overline{Sp}\{e_n\} = X \Rightarrow \overline{Sp}\{e_n\}^{\perp} = X^{\perp} = \{0\}.$

Prop: $\{e_k\}$ is complete iff for all $x \in X$,

$$\|x\|^2 = \sum |\langle x, e_k\rangle|^2. \longleftarrow \text{Parsevals Identity}$$

Pf: If not complete, $\exists\ x,\ x \neq 0,\ \langle x, e_k\rangle = 0\ \forall k.$

So $\|x\|^2 \neq \sum |\langle x, e_k\rangle|^2.$

Suppose $\{e_k\}$ is complete so

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, e_k\rangle e_k.$$

Then $\|x\|^2 = \| \lim \text{---} \|^2$

$$= \lim_{n\to\infty} \left\|\sum_{k=1}^{n} \langle x, e_k\rangle e_k\right\|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} |\langle x, e_k\rangle|^2$$

$$= \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2.$$

Bessel: $\sum |\langle x, e_k \rangle|^2 \leq \|x\|^2$ generally

Parseval: $\sum |\langle x, e_k \rangle|^2 = \|x\|^2$ for a <u>complete</u> o.n. sequence.

Def: An o.n. basis for a Hilbert space is a complete o.n. sequence.

Prop: A infinite-dim Hilbert space is separable iff it admits an o.n. basis.

Sketch: separable: $\{x_n\}$ dense.

reduce to a lin ind set. ; the span is still dense

Perform Gram-Schmidt. The span is the same, and hence dense. $\Longrightarrow$ complete

If $\{e_k\}$ is an o.n. basis, I claim

$$\{x : \underset{\exists n,}{\uparrow} \sum_{k=1}^{n} q_k e_k \quad q_k \in \mathbb{Q}\} \text{ is dense.}$$