The three examples show not closed or not convex or not Hilbert and the result can fail.

Thm: If A is a closed convex subset of a Hilbert space X, given $x \in X$ there exists a unique $a \in A$, $d(a,x) = d(A,x)$

Pf: Let a_n be a sequence in A, $d(a_n,x) \to d(a,x)$.

$(d(a_n,x) \leq d(a,x) + \frac{1}{n})$

By the parallelogram law, $\forall$ n, m

$$\|(p-a_n)-(p-a_m)\|^2 + \|(p-a_n)+(p-a_m)\|^2$$

$$\leq 2\|p-a_m\|^2 + 2\|p-a_n\|^2$$

Now $$\|(p-a_n)+(p-a_m)\|^2 = \|2\left(p-\left(\tfrac{a_n+a_m}{2}\right)\right)\|^2$$

$$= 4\|p-\left(\tfrac{a_n+a_m}{2}\right)\|^2$$

$$\geq 4\,d(p,A)$$ since A is convex.

Thus $\|a_n - a_m\|^2 \leq 4(d(p,A)) + \frac{1}{n} + \frac{1}{m} - 4d(p,A)$.

Hence $\{a_n\}$ is Cauchy and converges to a limit a.

Since A is closed, $a \in A$.

Moreover $d(p,A) \leq d(p,a) = \lim d(p,a_n) = d(p,A)$.

This establishes existence.

For uniqueness, if a_1 and a_2 are minimizers

$$\|(p-a_1)+(p-a_2)\|^2 + \|(p-a_1)-(p-a_2)\|^2$$

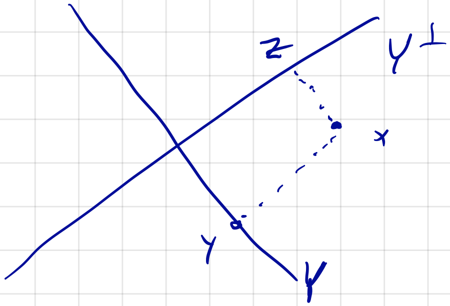
$$\leq 2\|p-a_1\|^2 + 2\|p-a_2\|^2$$

I.e. $\underbrace{4\|p-(\tfrac{a_1+a_2}{2})\|^2} + \|a_1-a_2\|^2 \leq 4\,d(p,A)$.

$4\,d(p,A) \leq$

so $\|a_1-a_2\|^2 \leq 0$.

Next up:



$x = y + z$

Given a subspace Y and $x \in X$ we would like to decompose

$$x = y + z \qquad \begin{array}{l} y \in Y \\ z \in Y^\perp. \end{array}$$

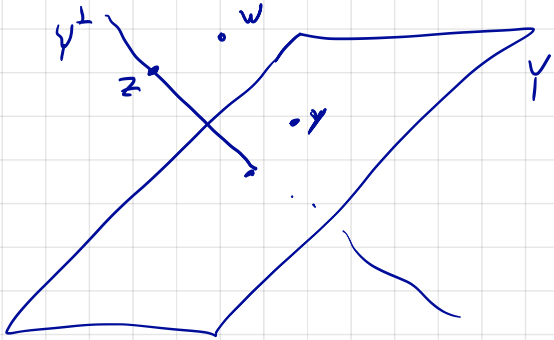
This doesn't always work: $Z \subseteq \ell^2$

$Z^\perp = \{0\}$

If $z \in Z$ and $w \in Z^\perp$ $w = 0$ so $z + w = z \in Z$.

The key extra ingredient is that Y must be closed.

Before we start:



$\|z\|^2$ vs $\|z-y\|^2$

$\le$? (compare vs w!)

Lemma: If $z \in Y^\perp$ then $\|z\| \le \|z-y\|$ for all $y \in Y$.

Pf:

$$\|z-y\|^2 = \|z\|^2 - \langle z,y\rangle - \langle y,z\rangle - \|y\|^2$$

$$= \|z\|^2 - \|y\|^2 \le \|z\|^2$$

0 is the closest point!

Converse:

If $\|z-y\|^2 \geq \|z\|^2$ for all $y \in Y$, $z \in Y^\perp$

$$\|z-\alpha y\|^2 = \|z\|^2 - \bar{\alpha}\langle z,y\rangle - \alpha\langle y,z\rangle + |\alpha|^2\|y\|^2 \le \|z\|^2$$

So $|\alpha|^2\|y\|^2 \le 2\,\mathrm{Re}\left[\langle y,z\rangle \alpha\right]$

Pick y so $\langle z,y\rangle \neq 0$, and pick α_0

so $\langle z,y\rangle \alpha_0 = 1$.

Then for $\alpha = t\alpha_0$, $t>0$

$$t^2 |\alpha_0|^2 \|y\|^2 \le 2t$$

But then $|\alpha_0|^2\|y\|^2 \le 2/t$ $\forall t>0$, and $\|y\|^2 = 0$,

a contradiction.

If $\|z\|^2 \le \|z-y\|^2$ $\forall y \in Y$ then $z \in Y^\perp$.

If not, we can find $y \in Y$, $\mathrm{Re}\langle y,z\rangle \neq 0$. For all $\alpha > 0$

$$\|z\|^2 \le \|z-\alpha y\|^2 = \|z\|^2 - 2\,\mathrm{Re}\left[\alpha\langle z,y\rangle\right] + |\alpha|^2\|y\|^2$$

So $\quad 2\,\mathrm{Re}\left[\alpha\langle z,y\rangle\right] \le |\alpha|^2\|y\|^2$

I.e. $\quad 2\,\mathrm{Re}\langle z,y\rangle \le \alpha\|y\|^2$.

But this is false for α suff small.

Thm: If $Y \in X$ is a closed subspace of a Hilbert space, given $x \in X$ there exists a unique $y \in Y$ and $z \in Y^\perp$,

$$x = y + z$$

Pf: Let $x \in X$. Since Y is closed and convex, there exists $y \in Y$, $d(x,y) = d(x,Y)$.

Let $z = x - y$.

I claim $z \in Y^\perp$.

Indeed if $a \in Y$,

$$\|z - a\| = \|\cancel{x} - y - a\| \geq d(x, Y) = \|x - y\| = \|z\|$$

Thus $z \in Y^\perp$.

As for uniqueness: $\cancel{x} = y_1 + z_1$
$x = y_2 + z_2$

$$\underbrace{y_2 - y_1}_{\in Y} = \underbrace{z_1 - z_2}_{\in Y^\perp}$$

But $Y \cap Y^\perp = 0$ $(\langle x, x \rangle = 0 !)$

So $z_1 = z_2$, $y_1 = y_2$.

Remark: if $a \perp b$, $\|a+b\|^2 = \|a\|^2 + \|b\|^2$

$\Downarrow$

$\langle a, b \rangle + \langle b, a \rangle = 0.$

So if $x = y + z$ $\quad y \in Y$, $z \in Y^\perp$,

$$\|x\|^2 = \|y\|^2 + \|z\|^2$$

Remark: For $(A^\perp)^\perp = A$ it must be

a) A is a subspace (Anything$^\perp$ is)
b) A is closed (— — —).

Prop: In a Hilbert space, $(Y^{\perp})^{\perp} = Y$ iff Y is a closed subspace.

Pf: Suppose Y is a closed subspace. Suppose $x \in (Y^{\perp})^{\perp}$. Let us write

$$x = y + z \quad \text{with } y \in Y \text{ and } z \in Y^{\perp}.$$ We claim $z = 0$.

Now $x \in (Y^{\perp})^{\perp}$ so

$$\langle x, z \rangle = 0.$$

But $\langle x, z \rangle = \langle y + z, z \rangle = \langle z, z \rangle = \|z\|^2$.

So $z = 0$ and $x \in Y$. That is $(Y^{\perp})^{\perp} \subseteq Y$.

But $Y \subseteq Y^{\perp\perp}$ always.

Other direction is obvious.

Bessel's Ineq:

$x \in X$

$$\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \qquad \text{converges?}$$

$$\langle y_n, y_n \rangle = \sum |\langle x, e_k \rangle|^2$$

Well,

$$y_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$$

$$\|x - y_n\|^2 = \|x\|^2 - \underbrace{\langle x, y_n \rangle - \langle y_n, x \rangle}_{2\,\mathrm{Re}\,\langle y_n, x \rangle} + \|y_n\|^2$$

But

$$\langle y_n, x \rangle = \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, x \rangle$$

$$= \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 = \|y_n\|^2 \in \mathbb{R}$$

So $\langle x, y_n \rangle$ is same.

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \qquad \text{converges.}$$