As a consequence, the norm <u>determines</u> the inner product.

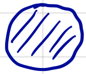
Inner products provide angles, too:

over $\mathbb{R}$:

$$-1 \le \underbrace{\frac{\langle x,y\rangle}{\|x\|\,\|y\|}}_{\cos\theta} \le 1 \qquad (x,y \neq 0)$$

$$\langle x,y\rangle = \|x\|\,\|y\|\cos\theta \qquad (!)$$

Over $\mathbb{C}$, not so clear since $\langle x,y\rangle$ is complex:



vs

Still

$\langle x,y\rangle = 0$ is special

$\cos\theta = 0$, $\theta = \frac{\pi}{2}$

We say vectors are orthogonal if $\langle x,y\rangle = 0$

Vectors $e_1, \dots, e_n$ in $X$ are orthonormal

if $\|e_i\|=1 \quad \forall i$

$$\langle e_i, e_j \rangle = 0 \quad i \neq j$$

$$\left( \langle e_i, e_j \rangle = \delta_{ij} \right)$$

Lemma: Any finite collection of orthonormal vectors is lin. i.d. ($e_1, \dots, e_n$)

Pf:

Suppose $c_1 e_1 + \dots + c_n e_n = 0$.

Then $0 = \langle c_1 e_1 + \dots + c_n e_n, e_k \rangle = \sum_{j=1}^{n} c_j \langle e_j, e_k \rangle$

$= c_k$

for each $k$.

Hence $e_1, \dots, e_n$ are a basis for their span.

If $x \in S = \text{span}(e_1, \dots, e_n)$

$$x = c_1 e_1 + \dots c_n e_n$$

$$\langle x, e_k \rangle = c_k .$$

That is, if I give you $x$, you know

$c_k$ via $\langle x, e_k \rangle$.

Compare: $w_1 = (-1, 2, 1)$
$w_2 = (2, 3, 5)$
$w_3 = (3, -1, 4)$

$x = (5, 0, 9)$

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & -1 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 9 \end{bmatrix}$$

Thm: Every finite dimensional vector space admits an o.n. basis.

PF: Gram-Schmidt

Start w/ a basis: $\rightarrow x_1, \dots, x_n$

$e_1 = \frac{x_1}{\|x_1\|}$

$f_2 = x_2 - \langle x_2, e_1 \rangle e_1$ $\quad \langle f_2, e_1 \rangle = \langle x_2, e_1 \rangle - \langle x_2, e_1 \rangle e_1$

$e_2 = f_2 / \|f_2\|$

$$f_k = x_k - \langle x_k, e_1\rangle e_1 - \cdots - \langle x_k, e_{k-1}\rangle e_{k-1}$$

$$e_k = f_k / \|f\|_k.$$

At each stage $f_k \neq 0$ (else $x_k$ is a linear combo of $e_1, \ldots, e_{k-1}$
and of $x_1, \ldots, x_{k-1}$)

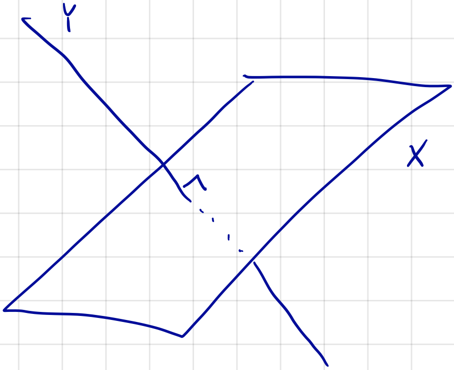
and $\langle e_k, e_j\rangle = 0 \quad k > j.$

---

Result is $n$ o.n. vectors. Lin ind + right count $\Rightarrow$ is a basis.

---

Consider the configuration:




The vectors in Y are all ortho to X.

and indeed it is all of such vectors.

Def: Given $A \subseteq X$, any subset, the orthogonal complement of A is: $A^\perp = \{ x \in X : \langle x, a \rangle = 0 \ \forall a \in A \}$.

(if $A = \phi$ we take $A^\perp = X$)

Observations

1) $A^\perp$ (the ortho complement of A) is a subspace even if A is not.

2) $0 \in A^\perp$ always.

3) If $A \subseteq B$, $B^\perp \subseteq A^\perp$

4) $(A^\perp)^\perp \supseteq A$, but need not equal A. even if A is a subspace.

5 $A^\perp$ is closed.

$x_n$ in $A^\perp$, $x_n \to x$

Fix $a \in A$. I claim $\langle x, a\rangle = \lim \langle x_n, a\rangle = 0$.

In fact $\begin{matrix} x_n \to x \\ y_n \to y \end{matrix} \Rightarrow \langle x_n, y_n\rangle \to \langle x, y\rangle$

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leq \|x_n\| \, \|y_n - y\| + \|x_n - x\| \, \|y\| \end{aligned}$$

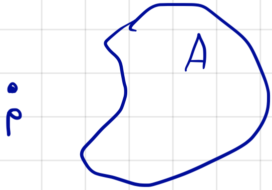
6) If $A$ contains an open ball, $A^\perp = 0$.

( If $y \in A^\perp$, $y \neq 0$ and if $B_r(a) \subseteq A$,

$$0 = \langle y, \underbrace{a + \frac{r}{2}\frac{y}{\|y\|}}_{\in A} \rangle = \langle y, \overbrace{a}^{\in A} \rangle + \frac{r}{2}\|y\|^2$$

$$= \frac{r}{2}\|y\|^2. \quad \text{oops!}$$

Distance from a set




$$d(p,A) = \inf_{a \in A} d(p,a)$$

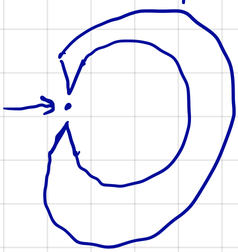
This distance always exists.
But it may not be that $\exists\, a \in A$,

$$d(p,A) = d(p,a)$$

e.g.

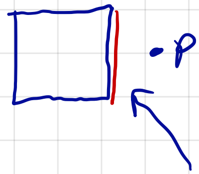

$d(p,A) = 0$ but if $d(p,a) = 0$, $a = p$.

Sometimes closest points exist, but are not unique



Fault of the weird shape.

But also:



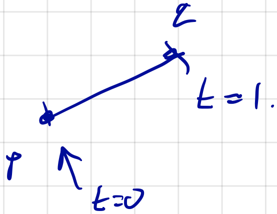

in $\ell^\infty$ norm, all these points are distance 1 from $p$.

**Result:** if $A \subseteq X$ is <u>convex</u> and $X$ is a Hilbert space, and <u>closed</u>

then closest points exist and are unique.

**Def** $A \subseteq X$ is <u>convex</u> if for any $p, q \in A$

$$(1-t)p + tq \in A \quad \forall\, t \in [0,1]$$



The three examples show not closed or not convex or not Hilbert and the result can fail.

Thm: If $A$ is a closed convex subset of a Hilbert space $X$, given $x \in X$ there exists a unique $a \in A$, $d(a,x) = d(A,x)$

Pf: Let $a_n$ be a sequence in $A$, $d(a_n,x) \to d(a,x)$.
$(d(a_n,x) \leq d(a,x) + \frac{1}{n})$

By the parallelogram law, $\forall n, m$

$$\|(p-a_n)-(p-a_m)\|^2 + \|(p-a_n)+(p-a_m)\|^2$$
$$\leq 2\|p-a_m\|^2 + 2\|p-a_n\|^2$$

Now $$\|(p-a_n)+(p-a_m)\|^2 = \|2(p-(\tfrac{a_n+a_m}{2}))\|^2$$
$$= 4\|p-(\tfrac{a_n+a_m}{2})\|^2$$
$$\geq 4\,d(p,A)$$ since $A$ is convex.

Thus $\|a_n-a_m\|^2 \leq 4(d(p,A)) + \frac{1}{n} + \frac{1}{m} - 4d(p,A)$.

Hence $\{a_n\}$ is Cauchy and converges to a limit $a$.

Since $A$ is closed, $a \in A$.

Moreover $d(p,A) \le d(p,a) = \lim d(p,a_n) = d(p,A)$.

This establishes existence.

For uniqueness, if $a_1$ and $a_2$ are minimizers

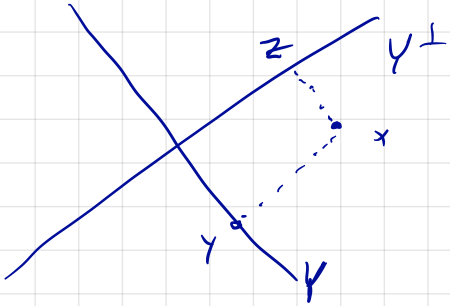
$$\|(p-a_1)+(p-a_2)\|^2 + \|(p-a_1)-(p-a_2)\|^2$$

$$\le 2\|p-a_1\|^2 + 2\|p-a_2\|^2$$

I.e. $\underbrace{4\left\|p-\left(\frac{a_1+a_2}{2}\right)\right\|^2}_{4d(p,A) \le} + \|a_1-a_2\|^2 \le 4\,d(p,A)$.

so $\|a_1-a_2\|^2 \le 0$.

Next up:




$x = y + z$

Given a subspace $Y$ and $x \in X$ we would like to decompose

$$x = y + z \qquad \begin{array}{l} y \in Y \\ z \in Y^\perp. \end{array}$$

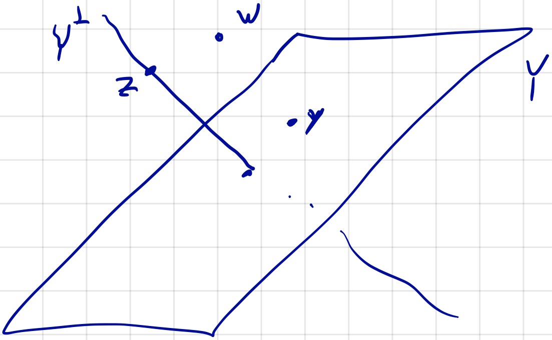
This doesn't always work: $Z \subseteq \ell^2$

$$Z^\perp = \{0\}$$

If $z \in Z$ and $w \in Z^\perp$ $w = 0$ so $z + w = z \in Z$.

The key extra ingredient is that $Y$ must be closed.

Before we start:




$\|z\|^2$ vs $\|z-y\|^2$

$\le ?$ (compare vs $w$!)

Lemma: If $z \in Y^\perp$ then $\|z\| \le \|z-y\|$ for all $y \in Y$.

Pf:

$$\|z-y\|^2 = \|z\|^2 - \langle z,y\rangle - \langle y,z\rangle - \|y\|^2$$

$$= \|z\|^2 - \|y\|^2 \le \|z\|^2.$$

Converse:

If $\|z-y\|^2 \le \|z\|^2$ for all $y \in Y$, $z \in Y^\perp$

$$\|z-\alpha y\|^2 = \|z\|^2 - \bar{\alpha}\langle z,y\rangle - \alpha\langle y,z\rangle + |\alpha|^2\|y\|^2 \le \|z\|^2$$

So $|\alpha|^2\|y\|^2 \le 2\,\mathrm{Re}\left[\langle y,z\rangle \alpha\right]$

Pick $y$ so $\langle z,y\rangle \neq 0$, and pick $\alpha_0$

so $\langle z,y\rangle \alpha_0 = 1$.

Then for $\alpha = t\alpha_0$ $t>0$

$$t^2 |\alpha_0|^2 \|y\|^2 \leq 2t$$

But then $|\alpha_0|^2 \|y\|^2 \leq 2/t$ $\forall\, t>0$, and $\|y\|^2 = 0$,

a contradiction.

Thm: If $Y \in X$ is a closed subspace of a Hilbert space, given $x \in X$ there exists a unique $y \in Y$ and $z \in Y^{\perp}$,

$$x = y + z$$

Pf: Let $x \in X$. Since $Y$ is closed and convex, there exists $y \in Y$, $d(x, y) = d(x, Y)$.

Let $z = x - y$.

I claim $z \in Y^{\perp}$.

Indeed if $a \in Y$,

$$\|z - a\| = \|\cancel{x} - y - a\| \geq d(x, Y) = \|x - y\| = \|z\|$$

Thus $z \in Y^{\perp}$.

As for uniqueness:
$$\cancel{x} = y_1 + z_1$$
$$x = y_2 + z_2$$

$$\underbrace{y_2 - y_1}_{\in Y} = \underbrace{z_1 - z_2}_{\in Y^{\perp}}$$

But $Y \cap Y^\perp = 0$ $\left( \langle x, x \rangle = 0 ! \right)$

So $z_1 = z_2$, $y_1 = y_2$.

---

Remark: if $a \perp b$, $\|a+b\|^2 = \|a\|^2 + \|b\|^2$

$$\Downarrow$$

$\langle a, b \rangle + \langle b, a \rangle = 0$.

So if $x = y + z$ $y \in Y$, $z \in Y^\perp$,

$$\|x\|^2 = \|y\|^2 + \|z\|^2$$