Last class: Facts about subspaces:

a) If X is a Banach space, $S \subseteq X$ is a Banach space $\Leftrightarrow$ it is closed.

b) If S is a subspace, so is $\overline{S}$

c) If X is a n.v.s and $S \subseteq$ is a subspace, if it is complete, then it is closed.

From c), if $S \subseteq X$ is finite dim, it is closed:

$\overbrace{\text{finite dim} \Rightarrow \text{complete}}^{\text{by 2 classes ago}} \Rightarrow \text{closed}$

$\underbrace{\text{complete} \Rightarrow \text{closed}}_{\text{by c)}}$

Thm: Suppose X is a normed space and $Y \subseteq X$ is a closed subspace, $Y \neq X$.

Given $\alpha \in (0,1)$ there exists $x_\alpha \in X$ with

$\|x_\alpha\| = 1$ and $\|x_\alpha - y\| > \alpha$ $\forall$ $y \in Y$.

Almost optimal: $\|x-y\| \geq 1$ is ideal, but not typically attainable.

Pf: Pick $x \in X \setminus Y$.

Let $d = \inf_{y \in Y} \|x-y\|$. Observe $d > 0$, otherwise is $y_n \to x$.

Since $\alpha^{-1} > 1$, $\exists\, z \in Y$, $\|x-z\| < \alpha^{-1} d$. (z is an approx closest point.)

Let $x_\alpha = \dfrac{x-z}{\|x-z\|}$, so $\|x_\alpha\| = 1$.

If $y \in Y_1$,

$$\|x_\alpha - y\| = \left\| \frac{x-z}{\|x-z\|} - y \right\|$$

$$= \frac{1}{\|x-z\|} \| \underbrace{x - z - \|x-z\| y}_{\in Y_1} \|$$

$$> \frac{d}{\|x-z\|}$$

$$> \alpha.$$

Cor: If X is infinite dimensional, $K = \{x \in X : \|x\| = 1\}$ is not compact.

Pf: Let $x_1 \in K$.

Let $S_1 = \text{span}(x_1)$. Then S_1 is finite dimensional and hence closed. There is $x_2 \in X$, $\|x_2\| = 1$,

$d(x_2, S_1) \geq \frac{1}{2}$.

Let $S_2 = \text{span}(x_1, x_2)$. Then S_2 is finite dim and closed. There is $x_3 \in X$ $\|x_3\| = 1$, $d(x_3, S_2) \geq 1/2$.

Continuing inductively, $\{x_n\}$ has not Cauchy subseq: $\|x_n - x_m\| \geq \frac{1}{2}$, $n \neq m$.

Exercise: K is closed.

Exercise: closed subsets of compact spaces are cpct.

Cor: If X is inf dim,

$\{x : \|x\| \leq 1\}$ is not compact.

Pf: If it were, K would be a closed subset of a compact space, and have compact.!

Next up: If complete, absolute convergence $\Rightarrow$ convergence.

$$\sum_{n=1}^{\infty} x_n \quad \text{means} \quad \lim_{N \to \infty} \underbrace{\sum_{n=1}^{N} x_n}_{S_N, \text{ partial sums.}}$$

A series is abs. conv. if $\sum_{n=1}^{\infty} \|x_n\|$ converges.

(From calc, abs. conv $\Rightarrow$ conv).

Thm: Suppose X is a Banach space.

If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then so does $\sum_{n=1}^{\infty} x_n$.

Power: it reduces convergence in X (complicated!) to convergence in $\mathbb{R}$.

Let $S_n = \sum_{k=1}^{n} x_k$

Pf. Let $T_n = \sum_{k=1}^{n} \|x_k\|$.

Since $\sum_{k=1}^{\infty} \|x_k\|$ converges, $\{T_n\}_{n=1}^{\infty}$ is Cauchy.

Let $\varepsilon > 0$. There exists N such that if $m > n \geq N$

$$|T_m - T_n| < \varepsilon.$$

But for this same N, if $m > n \geq N$,

$$\|S_m - S_n\| = \left\|\sum_{k=n+1}^{m} x_k\right\| \leq \sum_{k=n+1}^{m} \|x_k\|$$
$$= |T_m - T_n|$$
$$< \varepsilon.$$

Thus $\{S_m\}$ is Cauchy. Since X is a Banach space, the sequence of partial sums converges.

In fact: converse also!

If X is a n.v.s and every abs conv ~~series~~ series converges, then X is complete!

$\boxed{\text{Guided HW?}}$

Banach spaces are the main players, but there is a subcategory that is especially important.

Range of spaces: ℓ^p $\quad 1 \leq p \leq \infty$

should be $0 \leq \frac{1}{p} \leq 1$

Mid point: $p = 2$, $\frac{1}{p} = \frac{1}{2}$.

These spaces have an extra structure, an inner product.

Recall: an inner product on a vector space X is

real

a map $\langle \cdot , \cdot \rangle : X \times X \to \mathbb{R}$

satisfying

a bilinear form

1) for all $y \in X$

$f(x) = \langle x, y \rangle$ is linear

$\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

2) for all $x \in X$ $g(y) = \langle x, y \rangle$ is linear

symmetric

3) $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in X$

positive def.

4) $\langle x, x \rangle \geq 0 \quad \forall x \in X$

5) $\langle x, x \rangle = 0 \iff x = 0$

An inner product is a symmetric, pos. def. bilinear form.

Over $\mathbb{C}$ the rule is a bit different.

$$\langle \cdot,\cdot\rangle : X \times X \to \mathbb{C}$$

$$\langle x,y\rangle = \overline{\langle y,x\rangle}$$

$$\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$$ like before

$$\langle x, \alpha y\rangle = \bar{\alpha}\langle x,y\rangle$$

e.g. on $\mathbb{R}^n$ $\langle x,y\rangle = x^T\cdot y = \sum_{k=1}^{n} x_k y_k$

on $\mathbb{C}^n$ $\langle x,y\rangle = \bar{y}^T\cdot x = \sum_{k=1}^{n} x_k \bar{y}_k$.

Given an inner product, we obtain a norm via

$$\|x\| = \sqrt{\langle x,x\rangle}$$

Hard part is Δ ineq.

$$\|x+\lambda y\| = \|x\|^2 + \langle x, \lambda y\rangle + \langle \lambda y, x\rangle + \langle \lambda y, \lambda y\rangle$$

$$= \|x\|^2 + 2\,\mathrm{Re}\left(\lambda \langle x,y\rangle\right) + |\lambda|^2 \|y\|^2$$

$$\leq \|x\|^2 + 2\left|\lambda\langle x,y\rangle\right| + |\lambda|^2\|y\|^2$$

$$= \|x\|^2 + 2|\lambda|\,|\langle x,y\rangle| + |\lambda|^2\|y\|^2$$

Now use a discriminant argument and

$\|x+\lambda y\| \geq 0 \Rightarrow$

$$|\langle x,y\rangle|^2 \leq \|x\|^2\|y\|^2$$

and

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|.$$

This is, again, the C-S inequality, for any inner product, complex or not.

Exercise: Show $\|\cdot\|$ is a norm. (Δ ineq via C-S).

---

e.g. $C[0,1]$ $\quad \langle f,g\rangle = \int_0^1 fg$.

Next HW: not complete!

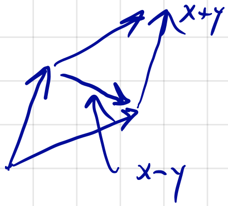
Def: A Hilbert space is an inner product space that is complete w.r.t. the induced norm.

---

important identities for i.p. spaces

1) parallelogram law

$$\|x+y\|^2+\|x-y\|^2 = 2\left(\|x\|^2+\|y\|^2\right)$$



(Sum of squares of four sides = sum of squares of diagonals")

Application: $\mathbb{R}^2$ with ℓ^1 norm is not an i.p. space.

$x = (1,-1)$

$y = (1,1)$

$\|x+y\|^2 = 4$

$\|x-y\|^2 = 4$

$\|x\|^2 = 4$

$\|y\|^2 = 4$

$$4+4 \neq 2(4+4)$$

b) polarization:

$(\mathbb{R})$ $\quad 4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2$

$(\mathbb{C})$ $\quad 4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\left[\|x+iy\|^2 - \|x-iy\|^2\right]$

As a consequence, the norm determines the inner product.