Claim:

$\mathbb{R}^n$ with $\|\cdot\|_\infty$ is complete

Pf: ($\mathbb{R}^2$) Suppose $\{p_n\}$ is Cauchy.

$$p_n = (x_n, y_n).$$

Observe $|x_n - x_m| \leq \|p_n - p_m\|_\infty$ and hence

$\{x_n\}$ is Cauchy in $\mathbb{R}$.

So $x_n \to x$ for some x.

Ditto, $y_n \to y$ for some y.

But if $p = (x, y)$,

$$0 \leq \|p - p_n\|_\infty \leq |x - x_n| + |y - y_n| .$$

Since $|x - x_n| \to 0$ and $|y - y_n| \to 0$, $\|p - p_n\| \to 0$

by the squeeze thm and $p_n \to p$.

Cor: $\mathbb{R}^n$ with <u>any</u> norm is complete.

Pf: Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and

$\{p_n\}$ is Cauchy with respect to it.

Since $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent,

$\{p_n\}$ is Cauchy with respect to $\|\cdot\|_\infty$.

Thus $p_n \to p$ for some p, w.r.t. $\|\cdot\|_\infty$.

But from equivalence of norms, $p_n \to p$ w.r.t. $\|\cdot\|$

as well.

Exercise (HW?) Any finite dim vector space is complete.

Def: A Banach space is a complete normed linear space.

All finite dimensional spaces are complete.

But: Z with l_∞ norm.

$$x_n = (1, \tfrac{1}{2}, \dots, \tfrac{1}{n}, 0, \dots)$$

$\|x_n - x_m\|_\infty \leq \min(\frac{1}{n}, \frac{1}{m})$, so is Cauchy.

But if $x = (x(1), \dots, x(N), 0, \dots 0)$

then $\|x - x_n\| \geq \frac{1}{N+1}$ for $n \geq N+1$.

So $x_n \not\to x$.

So the sequence has no limit.

If $S \subseteq X$ is a subspace, finite dim, then S is closed.

ℓ_1 is Complete

x_n Cauchy in ℓ_1

Given $\varepsilon > 0$ $\exists N$, $n, m \geq N \Rightarrow$

$$\|x_n - x_m\|_1 = \sum_{k=1}^{\infty} |x_n(k) - x_m(k)| < \varepsilon.$$

In particular, $n, m \geq N \Rightarrow |x_n(k) - x_m(k)| < \varepsilon$ if $n, m \geq N$.

I.e. for each k, $x_n(k)$ is Cauchy in $\mathbb{R}$ and converges to some limit $x(k)$.

Q: is $x = (x(1), \ldots)$ $\in \ell_1$?

Q: if so, does $x_n \to x$ in ℓ^1?

A: Cauchy sequences are bounded. So $\exists M$,

$$\sum_{k=1}^{\infty} |x_n(k)| \leq M \quad \forall n.$$

For each N, $\sum_{k=1}^{N} |x_n(k)| \leq M.$

$$\Rightarrow \sum_{k=1}^{N} |x(k)| \leq M \Rightarrow \sum_{k=1}^{\infty} |x(k)| \leq M.$$

Let $\varepsilon > 0$. Pick N so $n, m \geq N \Rightarrow \|x_m - x_n\|_1 < \varepsilon$

A:

If $n \geq N$,
$$\sum_{k=1}^{K} |x(k) - x_n(k)| = \lim_{m\to\infty} \sum_{k=1}^{K} |x_m(k) - x_n(k)|$$

But $\sum_{k=1}^{K} | \text{—} | \leq \|x_m - x_n\|_1 < \varepsilon$ if $m \geq N$.

So $\lim_{m\to\infty} \uparrow \leq \varepsilon$.

I.e $\sum_{k=1}^{K} |x(k) - x_n(k)| \leq \varepsilon$ if $n \geq N$, $\forall K$.

So $\sum_{k=1}^{\infty} |x(k) - x_n(k)| \leq \varepsilon$ if $n \geq N$.

So $\|x - x_n\|_1 \leq \varepsilon < 2\varepsilon$ if $n \geq N$.

$\boxed{\text{HW } \ell_\infty \text{ is complete}}$

If X is a Banach space and $S \subseteq X$ is a subspace,

S need not be complete. (!)

(e.g. $\mathbb{Z} \subseteq \ell_\infty$)

In fact:

Prop: A subspace of a Banach space is complete, and hence a Banach space, if and only if it is closed.

Pf: Let S be a subspace of a Banach space X.

Suppose S is complete. Let (x_n) be a sequence in S converges to $x \in X$. Then (x_n) is Cauchy in S and converges in S to some $y \in S$. But $x_n \to y$ in X as well. By uniqueness of limits $x = y \in S$. Thus S is closed.

Suppose S is closed. Let (x_n) be Cauchy in S and hence also in X. Then it converges in X

Does not depend as X is Banach

to some x. But S is closed and hence $x \in S$.

Thus $x_n \to x$ in S as well.

Remark: If X is any n.v.s. and $S \subseteq X$ is a finite dim vector space, then S is closed:

Finite dim $\Rightarrow$ complete $\Rightarrow$ closed by above.

What if S isn't closed? We can form $\overline{S}$. But is this a subspace?

Prop: If $S \subseteq X$ is a subspace, so is $\overline{S}$.

Pf: Suppose $\bar{x}, \bar{y} \in \overline{S}$. So there exist $(x_n), (y_n)$ in S, $x_n \to \bar{x}$, $y_n \to \bar{y}$.

Thus $(x_n + y_n) \to \bar{x} + \bar{y}$. But

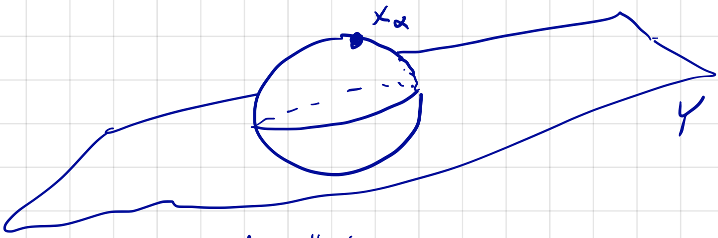
$x_n + y_n \in S$ $\forall n$, so $\bar{x} + \bar{y} \in \overline{S}$.

Ditto for scaling.

Thm: Suppose X is a normed space and $Y \subseteq X$ is a closed subspace, $Y \neq X$.

Given $\alpha \in (0,1)$ there exists $x_\alpha \in X$ with

$\|x_\alpha\| = 1$ and $\|x_\alpha - y\| > \alpha$ $\forall$ $y \in Y$.



Almost optimal: $\|x-y\| \geq 1$ is ideal, but not typically attainable.

Pf: Pick $x \in X \setminus Y$.

Let $d = \inf_{y \in Y} \|x-y\|$. Observe $d > 0$, otherwise is $y_n \to x$.

Since $\alpha^{-1} > 1$, $\exists\, z \in Y$, $\|x-z\| < \alpha^{-1} d$. (z is an approx closest point.)

Let $x_\alpha = \dfrac{x-z}{\|x-z\|}$, so $\|x_\alpha\| = 1$.

$$\|x_\alpha - y\| = \left\| \frac{x - z}{\|x - z} - y \right\|$$

$$= \frac{1}{\|x-z\|} \| x - \underbrace{z - \|x-z\| y}_{\in Y} \|$$

$$> \frac{d}{\|x-z\|}$$

$$> \alpha.$$

Cor: If X is infinite dimensional, $K = \{x \in X : \|x\| = 1\}$ is not compact.

Pf: Let $x_1 \in K$.

Let $S_1 = \operatorname{span}(x_1)$. Then S_1 is finite dimensional and hence closed. There is $x_2 \in X$, $\|x_2\| = 1$,

$$d(x_2, S_1) \geq \tfrac{1}{2}.$$

Let $S_2 = \operatorname{span}(x_1, x_2)$. Then S_2 is finite dim and closed. There is $x_3 \in X$ $\|x_3\| = 1$, $d(x_3, S_2) \geq 1/2$.

Continuing inductively, $\{x_n\}$ has not Cauchy subseq: $\|x_n - x_m\| \geq \frac{1}{2}$, $n \neq m$.

Cor: If X is inf dim,

$$\{x : \|x\| \leq 1\}$$ is not compact.

Pf: If it were, K would be a closed subset of a compact space, and hence compact.!

Next up: If complete, absolute convergence $\Rightarrow$ convergence.