Equivalent norms:

Consider $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$

$\|x\|_1 \leq 2\|x\|_\infty$ $\qquad$ $\|x\|_1 = |x_1| + |x_2|$

$\|x\|_\infty \leq \|x\|_1$ $\qquad$ $\|x\|_\infty = \max(|x_1|, |x_2|)$

$$\|x\|_\infty \leq \|x\|_1 \leq 2\|x\|_\infty$$

This configuration occurs more generally:

$\|\cdot\|_1$ and $\|\cdot\|_2$ on X are <u>equivalent</u> if $\exists\, m, M > 0$,

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

($m \leq M$, of course. $m = 1$, $M = 2$ in above)

$(X, \|\cdot\|_1) \sim (X, \|\cdot\|_2)$ if norms are equivalent.

Prop: This is an equivalence relation.

Pf:

$$m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1$$

$$n \|x\|_2 \leq \|x\|_3 \leq N \|x\|_2$$

$$mn \|x\|_1 \leq n\|x\|_2 \leq \|x\|_3 \leq N \|x\|_2 \leq NM \|x\|_1$$

The point:

Prop: If $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are equivalent then

$$x_k \to x \text{ w.r.t } \|\cdot\|_1 \iff x_k \to x \text{ w.r.t. } \|\cdot\|_2.$$

To this end:

Lemma: Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on X and $\exists K$,

$$\|x\|_1 \le K\|x\|_2 \quad \forall x \in X. \text{ Then}$$

a) If $x_k \to x$ w.r.t $\|\cdot\|_2$, $\Rightarrow$ $x_k \to x$ w.r.t $\|\cdot\|_1$.

b) If $\{x_k\}$ is Cauchy w.r.t. $\|\cdot\|_2$ $\Rightarrow$ $\{x_k\}$ is Cauchy w.r.t. $\|\cdot\|_1$

Pf of a):

Suppose $x_k \to x$. w.r.t $\|\cdot\|_2$.

Then $0 \le \|x - x_k\|_1 \le K\|x - x_k\|_2$.

Since $\lim \|x - x_k\|_2 = 0$, by the squeeze thm,

$\lim \|x - x_k\|_1 = 0$ also and $x_k \to x$ w.r.t. 1.

b) Let $\varepsilon > 0$. Pick N so if $n, m \ge N$, $\|x_n - x_m\|_2 < \frac{\varepsilon}{K}$.

Then if $n, m \ge N$, $\|x_n - x_m\|_1 \le K\|x_n - x_m\|_2 < K\frac{\varepsilon}{K} = \varepsilon$.

Execise: Now show convegace thm.

Big claim: All norms on a finite dim vector space are equivalent.

Lemma: Suppose $x_1, \dots, x_n$ is a basis for X.

Define $f: \mathbb{R}^n \to X$

$$f(\beta_1, \dots, \beta_n) = \|\beta_1 x_1 + \cdots + \beta_n x_n\|.$$

Then f is continuous ($\mathbb{R}^n$ with $\|\cdot\|_\infty$).

Pf: Let $M = \|x_1\| + \cdots + \|x_n\|$.

Given $\beta, \hat{\beta} \in \mathbb{R}^n$:

$$\begin{aligned} &|f(\beta_1, \dots, \beta_n) - f(\hat{\beta}_1, \dots, \hat{\beta}_n)| \\ &= |(\beta_1 - \hat{\beta}_1)x_1 + \cdots + (\beta_n - \hat{\beta}_n)x_n| \\ &\leq |\beta_1 - \hat{\beta}_1| \|x_1\| + \cdots + |\beta_n - \hat{\beta}_n| \|x_n\| \\ &\leq M \|\beta - \hat{\beta}\|_\infty . \end{aligned}$$

Thus if $\|\beta - \hat{\beta}\|_\infty < \frac{\varepsilon}{M}$, $|f(\beta) - f(\hat{\beta})| < \varepsilon$

and f is uniformly continuous. (Same δ!).

$S = \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$

I claim S is compact.

Idea ($\mathbb{R}^2$)

(x_k, y_k) in S

$|x_k| \leq 1$ $\quad$ $x_{k_j} \to x$

$|y_{k_j}| \leq 1$ $\quad$ $\hat{y}_\ell = y_{k_{j_\ell}} \to y$

$\hat{x}_\ell = x_{k_{j_\ell}} \to x$

$$\overset{\hat{p}_\ell}{(\hat{x}_\ell, \hat{y}_\ell)} \xrightarrow{\|\cdot\|_\infty} \overset{p}{(x, y)}$$

$$\left(\|\hat{p}_\ell - p_\ell\|_\infty \leq |\hat{x}_\ell - x| + |\hat{y}_\ell - y| \to 0. \right.$$

Finally, $\|\hat{p}_\ell\|_\infty = 1$ for all ℓ

$\|p\|_\infty = 1$ also. (Continuity of norm)

From the lemma, since S is cpct,

$$m \le f(\beta) \le M \qquad \text{for all } \beta \in S.$$

Moreover $\quad f(\lambda\beta) = \| \lambda\beta_1 x_1 + \cdots + \lambda \beta_n x_n \|$

$$= |\lambda| \, f(\beta)$$

$$m \, \|\beta\|_\infty \le f(\beta) \le M \, \|\beta\|_\infty \qquad \text{for all } \beta \in \mathbb{R}^n$$

Now with $\quad \underset{f_1}{\|\cdot\|_1}, \ \underset{f_2}{\|\cdot\|_2}$

$$m_1 \|\beta\|_\infty \le f_1(\beta) \le M_1 \|\beta\|_\infty$$

$$m_2 \|\beta\|_\infty \le f_2(\beta) \le M_2 \|\beta\|_\infty$$

Given $x \in X$, $\quad x = \beta_1 x_1 + \cdots + \beta_n x_n \quad$ for some β.

$$\|x\|_1 = f_1(\beta)$$

$$\|x\|_2 = f_2(\beta)$$

$$\|x\|_2 = f_2(\beta) \leq M_2 \|\beta\|_\infty = \frac{M_2}{m_1} m_1 \|\beta\|_\infty \leq \frac{M_2}{m_1} f_1(\beta) = \frac{M_2}{m_1} \|x\|_1$$

Similarly

$$\|x\|_1 \leq \frac{M_1}{m_2} \|x\|_2$$

$$\frac{m_2}{M_1} \|x\|_1 \leq \|x\|_2 \leq \frac{M_2}{m_1} \|x\|_1$$

So norms aren't so riveting on finite dimensional vector spaces
All alike!
No different notions of convergence.
No different notions of Cauchy sequences.

But: $Z \to$ sequences that end in zeros.

$\|\cdot\|_1$, $\|\cdot\|_\infty$ are norms on those spaces.

$(Z \subseteq \ell_1, \ Z \subseteq \ell_\infty)$.

These norms are not equivalent:

e.g: $z_k = (1, 1, \ldots, 1, 0, \ldots\ldots 0)$

$\|z_k\|_\infty = 1$

$\|z_k\|_1 = k$

If equiv, $\|z_k\|_1 \leq M \|z_k\|_1 = M$

$k \leq M \quad \forall k$ no!

Claim:

$\mathbb{R}^n$ with $\|\cdot\|_\infty$ is complete

Pf: $(\mathbb{R}^2)$ Suppose $\{p_n\}$ is Cauchy.

$$p_n = (x_n, y_n).$$

Observe $|x_n - x_m| \leq \|p_n - p_m\|_\infty$ and hence

$\{x_n\}$ is Cauchy in $\mathbb{R}$.

So $x_n \to x$ for some x.

Ditto, $y_n \to y$ for some y.

But if $p = (x, y)$,

$$0 \leq \|p - p_n\|_\infty \leq |x - x_n| + |y - y_n|.$$

Since $|x - x_n| \to 0$ and $|y - y_n| \to 0$, $\|p - p_n\| \to 0$

by the squeeze thm and $p_n \to p$.

Cor: $\mathbb{R}^n$ with <u>any</u> norm is complete.

Pf: Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $\{p_n\}$ is Cauchy with respect to it.

Since $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent, $\{p_n\}$ is Cauchy with respect to $\|\cdot\|_\infty$.

Thus $p_n \to p$ for some p, w.r.t. $\|\cdot\|_\infty$.

But from equivalence of norms, $p_n \to p$ w.r.t. $\|\cdot\|$ as well.

Exercise (HW?) Any finite dim vector space is complete.

Def: A <u>Banach</u> space is a <u>complete</u> normed linear space.

All finite dimensional spaces are complete.

But: Z with l_∞ norm.

$$x_n = (1, \tfrac{1}{2}, \ldots, \tfrac{1}{n}, 0, \ldots)$$

$$\|x_n - x_m\|_\infty \leq \min(\tfrac{1}{n}, \tfrac{1}{m}) \text{, so is Cauchy.}$$

But if $x = (x(1), \ldots, x(N), 0, \ldots 0)$

then $\|x - x_n\| \geq \frac{1}{N+1}$ for $n \geq N+1$.

So $x_n \not\to x$.

So the sequence has no limit.

ℓ_1 is Complete

x_n Cauchy in ℓ_1

Given $\varepsilon > 0$ $\exists$ N, $n, m \geq N \Rightarrow$

$$\|x_n - x_m\|_1 = \sum_{k=1}^{\infty} |x_n(k) - x_m(k)| < \varepsilon.$$

In particular, $n, m \geq N \Rightarrow |x_n(k) - x_m(k)| < \varepsilon$ if $n, m \geq N$.

I.e. for each k, $x_n(k)$ is Cauchy in $\mathbb{R}$ and converges to some limit $x(k)$.

Q: is $x = (x(1), \ldots)$ $\in \ell_1$?

Q: if so, does $x_n \to x$ in ℓ^1?

A: Cauchy sequences are bounded. So $\exists$ M,

$$\sum_{k=1}^{\infty} |x_n(k)| \leq M \quad \forall n.$$

For each N,

$$\sum_{k=1}^{N} |x_n(k)| \leq M.$$

$$\Rightarrow \sum_{k=1}^{N} |x(k)| \leq M \Rightarrow \sum_{k=1}^{\infty} |x(k)| \leq M.$$

Let $\varepsilon > 0$. Pick N so $n, m \geq N \Rightarrow \|x_m - x_n\|_1 < \varepsilon$

A:

If $n \geq N$, $$\sum_{k=1}^{K} |x(k) - x_n(k)| = \lim_{m\to\infty} \sum_{k=1}^{K} |x_m(k) - x_n(k)|$$

But $$\sum_{k=1}^{K} | \text{—} | \leq \|x_m - x_n\|_1 < \varepsilon \text{ if } m \geq N.$$

So $$\lim_{m\to\infty} \uparrow \leq \varepsilon.$$

So $$\sum_{k=1}^{K} |x(k) - x_n(k)| \leq \varepsilon \text{ if } n \geq N, \ \forall K.$$

So $$\sum_{k=1}^{\infty} |x(k) - x_n(k)) \leq \varepsilon \text{ if } n \geq N.$$

So $$\|x - x_n\|_1 \leq \varepsilon < 2\varepsilon \text{ if } n \geq N.$$