Why care about compact sets?

One reason:

If $A \subseteq X$ is compact and $f: A \to \mathbb{R}$ is continuous, then $f(A)$ is compact $\Rightarrow$ closed and bounded.

Bound above $\Rightarrow$ admits a supremum.

Closed $\Rightarrow$ $\sup(f(A)) \in f(A)$ (if $S \subseteq \mathbb{R}$ is a set, is a seq $\{x_n\}$ $x_n \to \sup S$, and now used closed).

$\Rightarrow \exists\, a_0 \in A,\ f(a_0) \geq f(a)\ \forall a \in A.$

f achieves a max!

ditto for a min.

A norm on a vector space is a function $X \to \mathbb{R}$

$\|x\|$

satisfying

1) $\|x\| \geq 0$, $\|x\| = 0 \iff x = 0$ $\forall x \in X$

2) $\|\alpha x\| = |\alpha| \, \|x\|$ $\forall \alpha \in \mathbb{F}, x \in X$

3) $\|x+y\| \leq \|x\| + \|y\|$.

Vector space + norm = "normed vector space"

From these, we get a metric:

$$d(x,y) = \|x-y\|.$$

This metric is compatible with v.s. operations:

$d(x+z, y+z) = d(x,y)$ (preserved under translation)

$$d(\alpha x, \alpha y) = \|\alpha(x-y)\| = |\alpha| \, \|x-y\|$$
$$= |\alpha| \, d(x,y)$$

Exercise: d is a norm

e.g: $\mathbb{R}^n$ $\quad \|x\| = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2}$

1), 2) trivial.

3):

Lemma: If $x, y \in \mathbb{R}^n$,

$$|x \cdot y| \leq \|x\| \, \|y\|$$

$\to$ Cauchy-Schwartz inequality

Pf: $\|x - \lambda y\|^2 = \|x\|^2 - 2\lambda x \cdot y + \lambda^2 \|y\|^2 \geq 0$

descriminant $\quad 4(x \cdot y)^2 - 4\|x\|^2 \|y\|^2 \leq 0$

$$\Rightarrow \quad |x \cdot y| \leq \|x\| \, \|y\|$$

and equality requires x, y are colinear.

Δ ineq:

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + 2x \cdot y + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

$$\Rightarrow \quad \|x+y\| \leq \|x\| + \|y\|$$

e.g (of normed vector space)

X cpct

$$C(X) = \{ f : X \to \mathbb{R} : f \text{ is cts} \}$$

$\mathbb{F}$

$$\|f\| = \max_{x \in X} |f(x)|$$

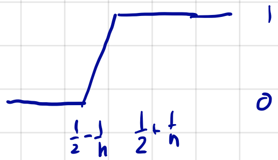
exercise: this is a norm.

e.g. $C[0,1]$

$$\|f\|_{L^2} = \left[\int_0^1 f^2 \right]^{1/2}$$

Exercise: Show $\int_0^1 fg \leq \|f\|_{L^2} \|g\|_{L^2}$ and establish the Δ ineq.

Alos, $C[0,1]$ is not complete with the L^2 norm.



$n \geq m$

$|f_n - f_m| = 0$ except on an interval of length $\frac{2}{n}$ where it is atmost 1.

$$\left[\int_0^1 |f_n - f_m|^2\right]^{1/2} \leq \left[\frac{2}{n}\right]^{1/2} \to 0.$$

Omit?

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e.g. $\ell^p \quad 1 \leq p \leq \infty$

sequences $\quad x = (x(1), \ldots)$

$$\|x\|_p = \left[\sum_{k=1}^{\infty} |x(k)|^p\right]^{1/p}$$

That the triangle ineq holds is a big deal (Minkowski's ineq).

ℓ^∞: bounded sequences $\quad \|x\|_\infty = \sup_k |x(k)|$

e.g: X a normed vector space

$S \subseteq X$ a subspace (closed under addition and scalar mult)

S inherits the norm.

e.g: $S = \{ x \in \ell^1 : \sum_{k=1}^{\infty} x(k) = 0 \}$

(note: $\sum_{k=1}^{\infty} |x(k)|$ exists, so $\sum_{k=1}^{\infty} x(k)$ converges)

e.g: X, Y normed vector spaces

$Z = X \times Y \longrightarrow$ (how is this a vector space?)

$\|(x,y)\| = \|x\|_X + \|y\|_Y$.

Exercise: $x_n \to x$ in $X \iff \lim_{n\to\infty} \|x_n - x\| = 0$

in $\mathbb{R}$!

Prop: In a n.v.s. X:

a) $\big| \|x\| - \|y\| \big| \le \|x-y\|$

b) $x_n \to x \implies \|x_n\| \to \|x\|$

c) If $x_n \to x$, $y_n \to y$ in X, $x_n + y_n \to x+y$

d) If $\alpha_n \to \alpha$ in $\mathbb{R}$, $x_n \to x$ in X, $\alpha_n x_n \to \alpha x$.

Pf d) $0 \le \| \alpha x - \alpha_n x_n \| = \| \alpha x - \alpha_n x + \alpha_n x - \alpha_n x_n \|$

$$\le \|(\alpha - \alpha_n) x\| + \|\alpha_n (x - x_n)\|$$

$$\le \underbrace{|\alpha - \alpha_n| \|x\|}_{\to 0} + \underbrace{|\alpha_n| \|x - x_n\|}_{\to 0}$$

So $\lim_{n\to\infty} \|\alpha x - \alpha_n x_n\| = 0$ by the squeeze Thm.

a) like metrics are cts, b) a consequence of metrics are cts.

Equivalent norms:

Consider $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$

$$\|x\|_1 \leq 2\,\|x\|_\infty \qquad \|x\|_1 = |x_1| + |x_2|$$

$$\|x\|_\infty \leq \|x\|_1 \qquad \|x\|_\infty = \max(|x_1|, |x_2|)$$

$$\|x\|_\infty \leq \|x\|_1 \leq 2\,\|x\|_\infty$$

This configuration occurs more generally:

$\|\cdot\|_1$ and $\|\cdot\|_2$ on X are <u>equivalent</u> if $\exists\, m, M > 0$,

$$m\,\|x\|_1 \leq \|x\|_2 \leq M\,\|x\|_1$$

($m \leq M$, of course. $m=1, M=2$ in above)

$(X, \|\cdot\|_1) \sim (X, \|\cdot\|_2)$ if norms are equivalent.

Prop: This is an equivalence relation.

Pf:

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1$$

$$n\|x\|_2 \le \|x\|_3 \le N\|x\|_2$$

$$mn\|x\|_1 \le n\|x\|_2 \le \|x\|_3 \le N\|x\|_2 \le NM\|x\|_1$$

The point:

Prop: If $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are equivalent then

$$x_k \to x \text{ w.r.t } \|\cdot\|_1 \iff x_k \to x \text{ w.r.t. } \|\cdot\|_2.$$

To this end:

Lemma: Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on X and $\exists K$,

$$\|x\|_1 \leq K \|x\|_2 \quad \forall x \in X. \text{ Then}$$

a) if $x_k \to x$ w.r.t $\|\cdot\|_2$, $\Rightarrow x_k \to x$ w.r.t $\|\cdot\|_1$.

b) if $\{x_k\}$ is Cauchy w.r.t. $\|\cdot\|_2 \Rightarrow \{x_k\}$ is Cauchy w.r.t. $\|\cdot\|_1$

Pf of a):

Suppose $x_k \to x$. w.r.t $\|\cdot\|_2$.

Then $0 \leq \|x - x_k\|_1 \leq K\|x - x_k\|_2$.

Since $\lim \|x - x_k\|_2 = 0$, by the squeeze thm,

$\lim \|x - x_k\|_1 = 0$ also and $x_k \to x$ w.r.t. 1.

b) Let $\varepsilon > 0$. Pick N so if $n,m \geq N$, $\|x_n - x_m\|_2 < \frac{\varepsilon}{K}$.

Then if $n,m \geq N$, $\|x_n - x_m\|_1 \leq K\|x_n - x_m\|_2 < K\frac{\varepsilon}{K} = \varepsilon$.

Exercise: Now show convergence thm.

Big claim: All norms on a finite dim vector space are equivalent.

Lemma: Suppose $x_1, \dots, x_n$ is a basis for X.

Define $f: \mathbb{R}^n \to X$

$$f(\beta_1, \dots, \beta_n) = \|\beta_1 x_1 + \cdots + \beta_n x_n\|.$$

Then f is continuous ($\mathbb{R}^n$ with $\|\cdot\|_\infty$).

Pf:

$$\left| f(\beta_1, \dots, \beta_n) - f(\hat{\beta}_1, \dots, \hat{\beta}_n) \right|$$

$$=$$