6) Open sets, closed sets $\rightsquigarrow$ as complement

7) point of closure $x_n \to x$

8 $\bar{A} = \bigcup$ of all points of closure

9) A is closed iff $A = \bar{A}$

10) Cts: $x_n \to x \Rightarrow f(x_n) \to f(x)$

(ε-δ, also)

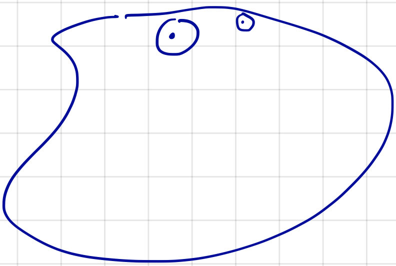
11) Cpct: seqs have conv sub seqs

12)

A set $U \subseteq X$ is <u>open</u> if $\forall x \in U$ $\exists r > 0$,

$B_r(x) \subseteq U$ $\boxed{B_x(r) \longrightarrow \text{wow}}$

r depends on x.



A set $A \subseteq X$ is closed if A^c is open.

$x \in X$ is a closure point of $A \subseteq X$, if there is a seq. $(x_n) \subseteq A$,

$$x_n \longrightarrow x.$$

$\bar{A}$ is the set of closure points of A.

$A \subseteq \bar{A}$: why?

Exercise: If A is closed, $\bar{A} \subseteq A$.

HW

Strategy: If $x \in A^c$, show x is not a point of closure

As a consequence $\bar{A} = A$ if A is closed.

Challenge: $\bar{A}$ is closed.

Suppose to contrary $\bar{A}^c$ is not open. So $\exists p$ for each n $B_{\frac{1}{n}}(p) \not\subseteq \bar{A}^c$.

So for each n $\exists$ $x \in B_{\frac{1}{n}}(p) \cap A$.

We will show $\bar{A}^c$ is open.

Suppose to produce a contradiction that $\bar{A}^c$ is not open.

Then there exists $p \in \bar{A}^c$ such that for all $\varepsilon > 0$, $B_\varepsilon(p) \not\subseteq \bar{A}^c$.

Thus for each $n \in \mathbb{N}$ we can find $\bar{a}_n \in \bar{A}$ with $\bar{a}_n \in B_{\frac{1}{2n}}(p)$.

But then, since $\bar{a}_n \in \bar{A}$ there is $a_n \in A$ with

$d(a_n, \bar{a}_n) < \frac{1}{2n}$. Notice, for each n,

$$d(a_n, p) \leq d(a_n, \bar{a}_n) + d(\bar{a}_n, a_n)$$
$$< \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

Thus, $\lim_{n \to \infty} d(a_n, p) = 0$ and $a_n \to p$.

I.e. $\{a_n\}$ is a seq in A converges to p. So $p \in \bar{A}$.

Yet $p \in \bar{A}^c$, a contradiction

Def: $f: X \to Y$ is continuous at $x \in X$, if

whenever $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y.

It is cts, if cts $\forall x$.

Thm: f is cts iff whenever $U \subseteq Y$ is open, $f^{-1}(U) \subseteq X$ is open.

$$f^{-1}(A^c) = f^{-1}(A)^c \quad \text{so also for closed!}$$

---

e.g. Fix $p \in X$. Define $f(x) = d(x,p)$, $f: X \to \mathbb{R}$.

Claim: f is cts. Fix x. Let $\varepsilon > 0$. Pick $\delta = \varepsilon$. If $d(x,z) < \delta$,

$$|f(x) - f(z)| = |d(x,p) - d(z,p)|$$

But
$$d(x,p) \leq d(x,z) + d(z,p) < \delta + d(z,p)$$
$$d(z,p) \leq d(x,z) + d(x,p) < \delta + d(x,p)$$

So

$$-\varepsilon = -\delta < d(x,p) - d(z,p) < \delta = \varepsilon.$$

I.e. $|d(x,p) - d(z,p)| < \varepsilon.$

Compact:

$A \subseteq X$ is compact if whenever $\{x_n\} \subseteq A$ is a sequence, it admits $\{x_{n_k}\}$, $x_{n_k} \to a \in A$ for some a.

Thm (Bolzano-Weierstrass)

$A \subseteq \mathbb{R}$ is compact $\Leftrightarrow$ it is closed and bounded.

If X is an arbitrary space and $A \subseteq X$ is compact, A is closed + bounded:

bounded: $\exists\ p, r \quad X \subseteq B_r(p)$.

Not bounded: $\forall p, r \ \exists\ x \in X, \ x \notin B_r(p)$.

Compact sets are bounded:

If not bounded, find p, x_n's $d(x_n, p) > n$.

If $x_{n_k} \to x$

$d(x_{n_k}, p) \to d(x, p)$ (use Δ ineq!)

But $d(x_{n_k}, p) \geq n_k \to \infty$.

Compact sets are closed:

Suppose x_n is a sequence in A, $x_n \to x$.
Need to show $x \in A$.

Is $\{x_{n_k}\}$, $x_{n_k} \to a \in A$.

But $x_{n_k} \to x$ (subseq of conv have same limit)

By uniqueness of limits, $x = a \in A$.

But converse is not true.

l_∞: set of bounded sequences

$x = (x(1), x(2), x(3), \dots)$

$d(x,y) = \sup_k \left(| x(k) - y(k) | \right)$

$x_1 = (1, 0, \dots)$
$x_2 = (0, 1, 0, \dots)$
$\vdots$

No Cauchy subseq, so no convergent either

No conv subsequence: $d(x_n, x_m) = 1$ $n \neq m$.
So no Cauchy subsequence.

Prop: If $A \subseteq X$ is compact and $f: X \to Y$ is cts,

$f(A)$ is compact.

Pf: Let $\{y_k\}$ be a sequence in $f(A)$.

$\forall k \ \exists \ x_k \in A,\ f(x_k) = y_k$.

By compactness of A, $\exists \ \{x_{k_j}\}$ $x_{k_j} \to a \in A$.

But then, by continuity $f(x_{k_j}) \to f(a)$.

That is, $y_{k_j} \to f(a) \in f(A)$.

Cor: If $f: X \to \mathbb{R}$ is continuous and X is compact, $\exists$ x_{min}, x_{max} such that

$$f(x_{min}) \leq f(x) \leq f(x_{max}) \quad \forall x \in X.$$

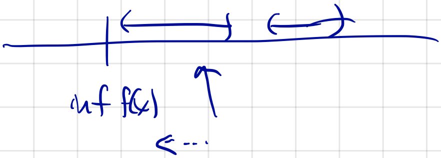
Pf: ~~Let $m = \inf f(X) \subseteq \mathbb{R}$; since $f(X)$ is bounded, m is finite, and since $f(X)$ is closed, $m \in f(X)$. Thus $\exists$ $x_m \in X$, $f(x_m) = m$. Evidently, $f(x_m) \leq f(x)$ $\forall x$.~~

~~Ditto for max.~~

Pf: Observe that X is closed and bounded.

Let $m = \inf f(X)$ ($m \le b \ \forall b \in f(X)$, and if any other $\hat{m}$ has this property $\hat{m} \le m$).

Let y_n be a sequence in $f(X)$ converging to $\inf f(X)$



For each n, pick x_n, $f(x_n) = y_n$.

Then $\{x_n\}$ has a subsequence in X, $x_{n_k} \to x$.
converging

but then $f(x_{n_k}) \to f(x)$. I.e. $y_{n_k} \to f(x)$.

But $y_{n_k} \to m$, and $f(x) = m$.

Lemma: If $f: X \to \mathbb{C}$ where X is cpct, there exists R such that

$$f(X) \subseteq B_R(0).$$

Pf: Compact sets are bounded.

X compact

$C_{\mathbb{F}}(X)$ $\quad \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ $\quad$ metric space.

Idea: First show is a vector space. So $f-g \in C_{\mathbb{F}}(X)$ if f, g are.

Then:

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

$f-g$ is cts!

Δ ineq:

For any x, $\quad |f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$

$$\leq d(f,h)$$

Now take as sup!

A norm on a vector space is a function $X \to \mathbb{R}$

$\|x\|$

satisfying

1) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$ $\forall x \in X$

2) $\|\alpha x\| = |\alpha| \, \|x\|$ $\forall \alpha \in F, \ x \in X$

3) $\|x+y\| \leq \|x\| + \|y\|$.

From these, we set a metric:

$$d(x,y) = \|x-y\|.$$

This metric is compatible with v.s. operations:

$$d(x+z, y+z) = d(x,y)$$ (preserved under translation)

$$d(\alpha x, \alpha y) = \|\alpha(x-y)\| = |\alpha| \, \|x-y\|$$

$$= |\alpha| \, d(x,y)$$

Exercise: d is a norm

e.g: $\mathbb{R}^n$ $\quad \|x\| = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2}$

1), 2) trivial.

3):

Lemma: If $x, y \in \mathbb{R}^n$,

$$|x \cdot y| \leq \|x\| \, \|y\|$$

Pf: $\|x - \lambda y\|^2 = \|x\|^2 - 2\lambda \, x \cdot y + \lambda^2 \|y\|^2 \geq 0$

descriminant ≤ 0: $4(x \cdot y)^2 + 4\|x\|^2 \|y\|^2$

$\Rightarrow \quad |x \cdot y| >$