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A sequence in a metric space is $\{x_k\}_{k=1}^{\infty}$, $x_k \in X$ $\forall k$.

($\mathbb{N} \to X$, formally)

A distance lets you detect if sequences converge.

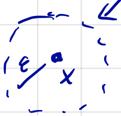
Def: $\{x_k\}$ converges to x $(x_k \to x)$

$$\lim_{k \to \infty} x_k = x$$

if $\forall \varepsilon > 0$ $\exists K$ such that if $k \geq K$,

$d(x_k, x) < \varepsilon$.

$B_\varepsilon(x) = \{y : d(x,y) < \varepsilon\}$



For each choice of $\varepsilon > 0$, you get trapped.

e.g. $\left(2^{-k}\sin(k),\ 2^{-k}\cos(k)\right) = x_k \in \mathbb{R}^2$

$$d(x_k, 0) = 2^{-k}$$

Given $\varepsilon > 0$, pick K so small so that $2^{-K} < \varepsilon$.
Then if $k \geq K$,

$$d(x_k, 0) = 2^{-k} \leq 2^{-K} < \varepsilon.$$

$$0.\overbrace{99\ldots9}^{n} \leq 1$$

$$\sum_{k=1}^{n} \frac{9}{10^k} = 9 \sum_{k=1}^{n} \frac{1}{10^k}$$

$$10 \cdot \sum_{k=1}^{n} 10^{-k} = \sum_{k=0}^{n-1} 10^{-k} = \sum_{k=1}^{n} 10^{-k} + 1 - 10$$

Lemma: Limits are unique.

Pf: Suppose $x_n \to x$ and $x_n \to y$, with $x \neq y$.

to produce a contradiction

Let $\varepsilon = d(x,y) > 0$. Pick N_1 so that if $n \geq N_1$,

$d(x_n, x) < \frac{\varepsilon}{2}$

Pick N_2 so if $n \geq N_2$, $d(x_n, y) < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$.

Then

$$
\begin{aligned}
d(x,y) &\leq d(x, x_N) + d(x_N, y) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

But $d(x,y) = \varepsilon$, a cont.

Related notion: Cauchy sequences. "terms get closer and closer together"

$x_1 = 3.1$
$x_2 = 3.14$
$x_3 = 3.141$
$\vdots$

$$|x_n - x_m| \leq 10^{-n} \quad (n \leq m)$$

Def: Cauchy if $\forall \varepsilon > 0 \ \exists N$ such that if $n, m \geq N$ then $d(x_n, x_m) < \varepsilon$.

Let $\varepsilon > 0$. Pick N so $10^{-N} < \varepsilon$.

If $n, m \geq N$, $|x_n - x_m| \leq 10^{-n} \leq 10^{-N} < \varepsilon$.

Lemma: Convergent sequences are Cauchy.

Pf: Suppose $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon > 0$. Pick N so that if $n \geq N$, $d(x_n, x) < \frac{\varepsilon}{2}$. Then, if $n, m \geq N$,

$$d(x_n, x_m) < d(x_n, x) + d(x, x_m)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon.$$

Converse is not always true:

e.g. $X = (0,1)$ in $\mathbb{R}$, with usual norm

$x_n = \frac{1}{n} \quad n \geq 2$

$x_n \to 0$ in $\mathbb{R}$

$\Rightarrow$ Cauchy

but if $x_n \to x$ in $(0,1)$ it also converges in $\mathbb{R}$, which violates uniqueness of limits.

More pernicious: $\mathbb{Q}$ has the same problem.

$3, 3.1, 3.14, \ldots$ is Cauchy in $\mathbb{Q}$, but not convergent in $\mathbb{Q}$.

Critical concept: A metric space is <u>complete</u> if every Cauchy sequence in it converges.

Power: You can detect convergent sequences without knowing what the limit is!

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6) Open sets, closed sets
as complement

7) point of closure $x_n \to x$

8 $\bar{A} = \cup$ of all points of closure

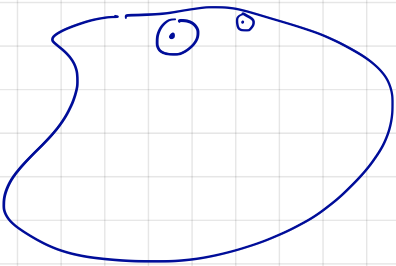
9) A is closed iff $A = \bar{A}$

10) Cts: $x_n \to x \Rightarrow f(x_n) \to f(x)$
(ε-δ, also)

11) Cpct: seqs have conv sub seqs

12)

A set $U \subseteq X$ is <u>open</u> if $\forall x \in U$ $\exists r > 0$,
$B_r(x) \subseteq U$ $\boxed{B_x(r) \longrightarrow \text{wow}}$
r depends on x.



A set $A \subseteq X$ is <u>closed</u> if A^c is open.

$x \in X$ is a closure point of $A \subseteq X$, if there is a seq. $(x_n) \subseteq A$,

$$x_n \longrightarrow x.$$

$\bar{A}$ is the set of closure points of A.

$A \subseteq \bar{A}$: why?

$$\boxed{\begin{array}{l} \text{Exercise: If } A \text{ is closed, } \bar{A} \subseteq A. \\ \text{HW} \qquad \text{Strategy: If } x \in A^c\text{, show } x \text{ is not a point of closure} \end{array}}$$

As a consequence $\bar{A} = A$ if A is closed.

Challenge: $\bar{A}$ is closed.

Suppose to contrary A^c is not open. So $\exists p$ for each n $B_{\frac{1}{n}}(p) \nsubseteq A^c$.

So for each n $\exists$ $x \in B_{\frac{1}{n}}(p) \cap A$.

Def: $f: X \to Y$ is continuous at $x \in X$, if

whenever $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y.

It is cts, if cts $\forall x$.

Thm: f is cts iff whenever $U \subseteq Y$ is open, $f^{-1}(U) \subseteq X$ is open.

$$f^{-1}(A^c) = f^{-1}(A)^c \quad \text{so also for closed!}$$

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e.g. Fix $p \in X$. Define $f(x) = d(x,p)$, $f: X \to \mathbb{R}$.

Claim: f is cts. Fix x. Let $\varepsilon > 0$. Pick $\delta = \varepsilon$. If $d(x,z) < \delta$,

$$|f(x) - f(z)| = |d(x,p) - d(z,p)|$$

But
$$d(x,p) \leq d(x,z) + d(z,p) < \delta + d(z,p)$$
$$d(z,p) \leq d(x,z) + d(x,p) < \delta + d(x,p)$$

So

$$-\varepsilon = -\delta < d(x,p) - d(z,p) < \delta = \varepsilon.$$

I.e. $|d(x,p) - d(z,p)| < \varepsilon.$

Compact:

$A \subseteq X$ is compact if whenever $\{x_n\} \subseteq A$ is a sequence, it admits $\{x_{n_k}\}$, $x_{n_k} \to a \in A$ for some a.

Thm (Bolzano-Weierstrass)

$A \subseteq \mathbb{R}$ is compact $\Leftrightarrow$ it is closed and bounded.

If X is an arbitrary space and $A \subseteq X$ is compact, A is closed + bounded:

bounded: $\exists\ p, r \quad X \subseteq B_r(p)$.

Not bounded: $\forall\ p, r\ \exists\ x \in X,\ x \notin B_r(p)$.

Compact sets are bounded:

If not bounded, find p, x_n's $d(x_n, p) > n$.

If $x_{n_k} \to x$

$d(x_{n_k}, p) \to d(x, p)$ (use Δ ineq!)

But $d(x_{n_k}, p) \geq n_k \to \infty$.

Compact sets are closed:

Suppose x_n is a sequence in A, $x_n \to x$.
Need to show $x \in A$.

Is $\{x_{n_k}\}$, $x_{n_k} \to a \in A$.

But $x_{n_k} \to x$ (subseq of conv have same limit)

By uniqueness of limits, $x = a \in A$.

But converse is not true.

l_∞: set of bounded sequences

$$x = (x(1), x(2), x(3), \dots)$$

$$d(x,y) = \sup_k \left(|x(k) - y(k)| \right)$$

$x_1 = (1, 0, \dots)$
$x_2 = (0, 1, 0, \dots)$
$\vdots$

No conv subsequence: $d(x_n, x_m) = 1$ $n \neq m$.
So no Cauchy subsequence.

Prop: If $A \subseteq X$ is compact and $f: X \to Y$ is cts,

$f(A)$ is compact.

Pf: Let $\{y_k\}$ be a sequence in $f(A)$.

$\forall k \ \exists \ x_k \in A, \ f(x_k) = y_k$.

By compactness of A, $\exists \ \{x_{k_j}\}$ $x_{k_j} \to a \in A$.

But then, by continuity $f(x_{k_j}) \to f(a)$.

That is, $y_{k_j} \to f(a) \in f(A)$.

Cor: If $f: X \to \mathbb{R}$ is continuous and X is compact, $\exists$ x_{min}, x_{max} such that

$$f(x_{min}) \leq f(x) \leq f(x_{max}) \quad \forall x \in X.$$

Pf: Let $m = \inf f(X) \subseteq \mathbb{R}$; since $f(X)$ is bounded, m is finite, and since $f(X)$ is closed, $m \in f(X)$. Thus $\exists \ x_m \in X$, $f(x_m) = m$. Evidently, $f(x_m) \leq f(x) \ \forall x$.

Ditto for max.

Lemma: If $f: X \to \mathbb{C}$ where X is cpct, there exists R such that

$$f(X) \subseteq B_R(0).$$

Pf: Compact sets are bounded.

X compact

$C_{\mathbb{F}}(X)$ $\quad \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ $\quad$ metric space.

Idea: First show is a vector space. So $f-g \in C_{\mathbb{F}}(X)$ if, f, g are.

Then:

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

$f-g$ is cts!

Δ ineq:

For any x, $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$

$$\leq d(f,h)$$

Now take as up!