Functional analysis concerns, in the beginning, linear algebra on infinite dimensional vector spaces.

Why do we care?

$$\ddot{x}(t) + \sin(t)\dot{x}(t) + t^2 x(t) = e^{-t^2}$$

$x(0) = 1$

$\dot{x}(0) = -3$

Is there a solution?

Here's the big vector space: functions $x : [-T, T] \to \mathbb{R}$, say, that are twice ctsly differentiable.

You can add them! $x(t) = x_1(t) + x_2(t)$

You can multiply by a number $x_\alpha(t) = \alpha x(t)$

This is the defining feature of a vector space!

$C^2[-T, T]$

$$L : C^2[-T, T] \to C^0[-T, T] \times \mathbb{R} \times \mathbb{R}$$

$$L(x) = \begin{cases} \ddot{x} + \sin(t)\dot{x} + t^2 x \\ \dot{x}(0) \\ x(0) \end{cases}$$

So, the problem of finding a solution of this <u>linear</u> ODE reduces to finding the solution of

$$L(x) = (e^{-t^2}, 1, -3)$$

L is a linear map: $L(x+y) = (\ddot{x} + \ddot{y} + \sin(t)(\dot{x}+\dot{y}) + t^2(x+y), \dot{x}(0)+\dot{y}(0), x+y)$

$$= L(x) + L(y)$$

$$L(\alpha x) = \alpha L(x)$$

(This is what defines a linear map!)

$$L: X \to Y \qquad L(x_1 + x_2) = L(x_1) + L(x_2)$$
$$L(\alpha x) = \alpha L(x)$$

Why is $C^2[-T,T]$ ∞-dim?

What is the dimension of a vector space?

$x_1, \dots, x_n \in X$ are <u>linearly independent</u> if

whenever $\alpha_1 x_1 + \dots + \alpha_n x_n = 0$, $\alpha_1 = \dots = \alpha_n = 0$.

This works for infinite collections, too, with the caveat that we don't add ∞-ly many things (coming soon!)

$\{x_\beta\}_{\beta \in I}$ is l.i. if whenever $\sum_{\beta \in I} \alpha_\beta x_\beta = 0$, all $\alpha_\beta = 0$.

$\beta = 0$ for all but finitely many

A basis for a vector space is a collection $\{x_\beta\}$

that
1) spans i.e. any one is a finite linear combo
2) linearly independent.

A space is n-dim if it has a basis with n vectors.

e.g. $e_1 = (1,0,0)$ $e_2 = (0,1,0)$ $e_3 = (0,0,1)$ is standard basis for $\mathbb{R}^3$.

$x e_1 + y e_2 + z e_3 = (x,y,z)$, so spans.

if $= 0$, $(x,y,z) = (0,0,0)$ so $x = 0$, etc.

Fact (from LA)

If V is finite dimensional, of dimension n, then any linearly independent collection of vectors has at most n elements.

Otherwise:

$x_1, \dots, x_n$ a basis.

$y_1, \dots, y_{n+1}$ linearly independent

$$x_1 a_{11} + \cdots + x_n a_{n1} = y_1$$

$$\vdots$$

$$x_1 a_{1\,n+1} + \cdots + x_n a_{n\,n+1} = y_{n+1}$$

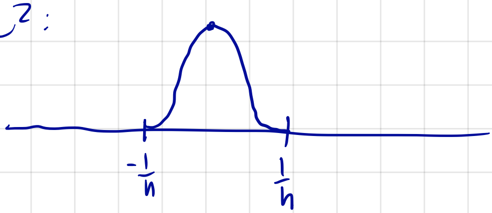
$A = [a_{ij}]$ is $n \times n+1$, has $As = 0$ $s \neq 0$

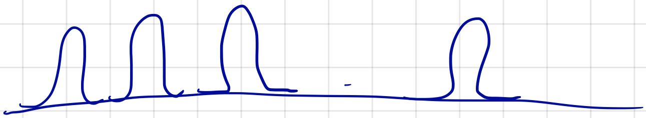
$$y_1 s_1 + \cdots + y_{n+1} s_{n+1} =$$

Summarize?

Read 1.1; ask two questions on Friday.

I can find C^2:





take n small, and get as many as I want.

We have functional analysis because we need to add infinitely many things:

e.g.: Fourier series

$$f(x) \text{"="} \sum_{k=1}^{\infty} \frac{1}{k} \cos(kx)$$

$X =$ 2π-periodic functions, cts?

= means the approximations $\sum_{k=1}^{N} \frac{1}{k^2} \cos(kx)$ get "better and better" as $N \to \infty$.

needs precision.

You might ask "for each x, $f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(kx)$".

But this won't let you do calculus:

$$f'(x) \stackrel{?}{=} \sum_{k=1}^{\infty} \frac{-1}{k} \sin(kx) \quad ??$$

Another distance: f_1 and f_2 are close if

$$\int_{-\pi}^{\pi} |f_1 - f_2|^2 \, dx \quad \text{is small} \quad (L^2)$$

or

$$\int_{-\pi}^{\pi} |f_1 - f_2| \, dx \quad \text{is small} \quad (L^1)$$

Genuinely different notions of distance

f_n

is possible.

$f_n \xrightarrow{L_1} 0$

$f_n \xrightarrow{L_2} 0$

Metric Spaces:

1) define
2) l_2, l_1, l_∞
3) convergent
4) Cauchy
5) limits are unique
   conv. $\Rightarrow$ Cauchy
   subseqs converge
6) Open sets, closed sets as complement
7) point of closure $x_n \to x$
8 $\overline{A} = \bigcup$ of all points of closure
9) A is closed iff $A = \overline{A}$
10) Cts: $x_n \to x \Rightarrow f(x_n) \to f(x)$
   (ε-δ, also)
11) Cpct: seqs have conv sub seqs
12)

Metric spaces:

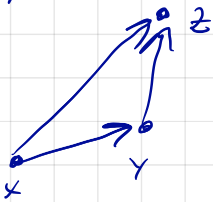
distance: $d : X \times X \to \mathbb{R}$.

$\forall x,y,z \in X$: $d(x,y) \geq 0$ $(=0 \iff x=y)$

$d(x,y) = d(y,x)$

$d(x,z) \leq d(x,y) + d(y,z)$

triangle inequality. allows length estimation.



$\mathbb{R}^n$: $d(x,y) = \left[\sum_{k=1}^{n} (x_k - y_k)^2 \right]^{1/2}$ ℓ_2 distance.

Euclidean,

alt: $d_1(x,y) = \sum_{k=1}^{n} |x_k - y_k|$ ℓ_1 distance.

satisfy? Hard part is Δ ineq. Try for d_1

A sequence in a metric space is $\{x_k\}_{k=1}^{\infty}$, $x_k \in X \ \forall k$.

($\mathbb{N} \to X$, formally)

A distance lets you detect if sequences converge.

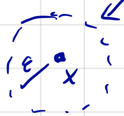
Def: $\{x_k\}$ converges to x $(x_k \to x)$

$$\lim_{k \to \infty} x_k = x$$

if $\forall \varepsilon > 0 \ \exists K$ such that if $k \geq K$,

$d(x_k, x) < \varepsilon$.

$B_\varepsilon(x) = \{y : d(x,y) < \varepsilon\}$



For each choice of $\varepsilon > 0$, you get trapped.

e.g. $\left(2^{-k}\sin(k), 2^{-k}\cos(k) \right) = x_k \in \mathbb{R}^2$

$$d(x_k, 0) = 2^{-k}$$

Given $\varepsilon > 0$, pick K so small so that $2^{-K} < \varepsilon$.
Then if $k \geq K$,

$$d(x_k, 0) = 2^{-k} \leq 2^{-K} < \varepsilon.$$

$$\overbrace{0.99\ldots\ldots9}^{n} \leq 1$$

$$\sum_{k=1}^{n} \frac{9}{10^k} = 9 \sum_{k=1}^{n} \frac{1}{10^k}$$

$$10 \cdot \sum_{k=1}^{n} 10^{-k} = \sum_{k=0}^{n-1} 10^{-k} = \sum_{k=1}^{n} 10^{-k} + 1 - 10$$

Lemma: Limits are unique.

Pf: Suppose $x_n \to x$ and $x_n \to y$, with $x \neq y$.

to produce a contradiction

Let $\varepsilon = d(x,y) > 0$. Pick N_1 so that if $n \geq N_1$,

$d(x_n, x) < \frac{\varepsilon}{2}$

Pick N_2 so if $n \geq N_2$, $d(x_n, y) < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$.

Then

$$\begin{aligned} d(x,y) &\leq d(x, x_N) + d(x_N, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

But $d(x,y) = \varepsilon$, a cont.

Related notion: Cauchy sequences. "terms get closer and closer together"

$x_1 = 3.1$
$x_2 = 3.14$
$x_3 = 3.141$
$\vdots$

$$|x_n - x_m| \leq 10^{-n} \quad (n \leq m)$$

Def: Cauchy if $\forall \varepsilon > 0 \ \exists N$ such that if $n, m \geq N$ then $d(x_n, x_m) < \varepsilon$.

Let $\varepsilon > 0$. Pick N so $10^{-N} < \varepsilon$.

If $n, m \geq N$, $|x_n - x_m| \leq 10^{-n} \leq 10^{-N} < \varepsilon$.

Lemma: Convergent sequences are Cauchy.

Pf: Suppose $\lim_{n \to \infty} x_n = x$.

Let $\varepsilon > 0$. Pick N so that if $n \geq N$, $d(x_n, x) < \frac{\varepsilon}{2}$. Then, if $n, m \geq N$,

$$d(x_n, x_m) < d(x_n, x) + d(x, x_m)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Converse is not always true:

e.g. $X = (0,1)$ in $\mathbb{R}$, with usual norm

$x_n = \frac{1}{n}$ $n \geq 2$

$x_n \to 0$ in $\mathbb{R}$

$\Rightarrow$ Cauchy

but if $x_n \to x$ in $(0,1)$ it also converges in $\mathbb{R}$, which violates uniqueness of limits.

More ~~ba~~ pathologies: $\mathbb{Q}$ has the same problem.

$3, 3.1, 3.14, \ldots$ is Cauchy in $\mathbb{Q}$, but not convergent in $\mathbb{Q}$.

Critical concept: A metric space is <u>complete</u> if every Cauchy sequence in it converges.