Mean Value Theorem. If f is a continuous function on an interval $[a, b]$ that has a derivative at every point in (a, b), then there is a point c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

1. Suppose f is a continuous function on $[a, b]$ that has a derivative at every point of (a, b). Suppose also that $f(b) \leq f(a)$. What can you conclude from the Mean Value Theorem?

2. Suppose f is a continuous function on $[a, b]$ that has a derivative at every point of (a, b), and that $f'(x) > 0$ for each x in (a, b). Thinking about your answer to problem 1, what can you conclude about $f(a)$ and $f(b)$?

3. A function is said to be **increasing** on an interval (a, b) if whenever x and z are in the interval and $x < z$, then $f(x) < f(z)$. It is **decreasing** if whenever x and z are in the interval and $x < z$, then $f(x) > f(z)$ Sketch an example of a function that is increasing on $(1, 3)$ and decreasing on $(3, 5)$.

Increasing/Decreasing Test

Your answer to problem 2 implies the first item below; the second is justified by a similar argument.

- If $f'(x) > 0$ on an interval (a, b) then f is increasing on the interval.
- If $f'(x) < 0$ on an interval (a, b) then f is decreasing on the interval.

4. Use the increasing/decreasing test to find intervals where

$$f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$$

is increasing and intervals where it is decreasing.

5. Find the critical points of the function $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$ from the previous problem. There should be two, c_1 and c_2 with $c_1 < c_2$. Just pay attention to c_1.

1. Just to the left of c_1 is the function increasing or decreasing?
2. Just to the right of c_1 is the function increasing or decreasing?
3. Now decide intuitively, based on these two observations, if f has a local min, local max, or neither at c_1.

6. Repeat the previous exercise for the other critical point c_2.

You have just sketched the argument that justifies the following:

First Derivative Test
Suppose f is a function with a derivative on (a, b), and if c is a point in the interval with $f'(c) = 0$.

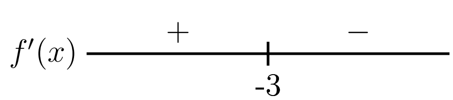
- If $f'(x) > 0$ for x just to the left of c and $f'(x) < 0$ for x just to the right of c, then f has a __________________ at c.

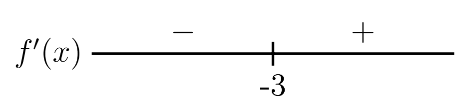
- If $f'(x) < 0$ for x just to the left of c and $f'(x) > 0$ for x just to the right of c, then f has a __________________ at c.

7. The function $f(x) = xe^x$ has exactly one critical point. Find it, and then use the First Derivative Test to determine if a local minimum or local maximum occurs there.

8. Consider the function $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$. Find intervals such that the **derivative** of $f(x)$ is increasing or decreasing.

9. Earlier you computed that $f'(-3) = 0$. Is f' increasing near $x = -3$ or decreasing near $x = -3$? Which of the following two scenarios must we have?





You have just sketched out justification for the following.

Second Derivative Test
Suppose f is a function with a continuous second derivative on (a, b), and that c is a point in the interval with $f'(c) = 0$.

- If $f''(c) > 0$ then f has a ____________________ at c.

- If $f''(c) < 0$ then f has a ____________________ at c.

10. Use the Second Derivative Test to determine if $f(x) = xe^x$ has a local min/max at its only critical point.

11. Consider the function $f(x) = x^3$. Verify that $f'(0) = 0$. Then decide what the Second Derivative Test has to say, if anything, about whether a local min/max occurs at $x = 0$.

First Derivative Test (Final Case)

- If $f'(c) = 0$ and $f'(x) < 0$ on both sides of c or $f'(x) > 0$ on both sides of c, then there is neither a local min nor a local max at c.

12. Decide what the First Derivative Test has to say, if anything, about whether a local min/max occurs at $x = 0$ for $f(x) = x^3$.