Linearization

Given a function $f(x)$, its linearization at $x = a$ is the function

$$L(x) = f(a) + f'(a)(x - a).$$

For example, if $f(x) = \sqrt{x}$ and $a = 4$ then $f(4) = 2$ and $f'(4) = 1/(2\sqrt{4}) = 1/4$. So

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

The graph of the linearization is just the tangent line to the curve $y = \sqrt{x}$ at $x = 4$. So we expect that $L(x)$ is a good approximation for $\sqrt{x}$ for x near 4. The point is that computing square roots is hard work (even if your calculator makes it look easy) but computing the value of a linear function like L is easy. In fact your calculator is doing a more sophisticated generalization of the linear approximation: stay tuned in Calculus II!

1. Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ to approxmiate $\sqrt{4.1}$ and compare your result to its approximation computed by your calculator.

2. Use the linear approximation to approximate the cosine of $29° = \frac{29}{30}\frac{\pi}{6}$ radians.

3. Find the linear approximation of $f(x) = \ln(x)$ at $a = 1$ and use it to approxmate $\ln(0.5)$ and $\ln(0.9)$. Compare your approximation with your calculator's. Sketch both the curve $y = \ln(x)$ and $y = L(x)$ and label the points $A = (0.5, \ln(0.5))$ and $B = (0.5, L(0.5)$

4. Find the linear approximation of $f(x) = e^x$ at $a = 0$ and use it to approximate $e^{0.05}$ and e^1 Compare your approximations with your calculator's.

Differentials Suppose we have a variable $y = f(x)$. We define its differential to be

$$dy = f'(x)dx$$

where x and dx are thought of as variables you can control. What's the point? The value of dy is an estimate of how much y changes if we change x into $x + dx$. See the graph:

5. A tree is growing and the radius of its trunk in centemeters is $r(t) = 2\sqrt{t}$ where t is measured in years. Use the differential to estimate the change in radius of the tree from 4 years to 4 years and one month.

6. A coat of paint of thinkness 0.05cm is being added to a hemispherical dome of radius 25m. Estimate the volume of paint needed to accomplish this task. [Challenge: will this be an underestimate or an overestimate? Thinking geometrically or thinking algebraically will both give you the same answer.]

7. The radius of a disc is 24cm with an error of ±0.5cm. Estimate the error in the area of the disc as an absolute and as a relative error.