Substitution:

Consider

$$\int \cos(x^3)\, 3x^2\, dx$$

From the chain rule

$$\frac{d}{dx} \sin(x^3) = \cos(x^3) \cdot 3x^2, \text{ so } \int \cos(x^3) 3x^2 dx = \sin(x^3)$$

The method of substitution runs the chain rule in reverse.

Mechanically: introduce a new variable $u = x^3$

$$du = 3x^2\, dx$$

just like differentials

Now convert all x's to u's

$$\int \cos(x^3)\, 3x^2 dx = \int \cos(u)\, du = \sin(u) = \sin(x^3) \checkmark$$

Why this works:

Consider $\int f'(g(x))\, g'(x)\, dx$

Let $u = g(x)$.

Replace, formally, $du = g'(x)\,dx$

A) $\int f'(g(x))\, \underbrace{g'(x)\, dx} = f(g(x))$ (use chain rule!)

B) $\int f'(u)\, du = f(u) = f(g(x))$

Same end result.

Let's practice.

$$\int \sin(3x+9)\,dx \qquad u = 3x+9$$

$$du = 3\,dx$$

$$\tfrac{1}{3}\,du = dx$$

$$\int \sin(u)\,\tfrac{1}{3}\,du$$

notice: all u's, no x's!

Convert everything!

$$\int \sin(3x+9)\,dx = \int \sin(u)\,\frac{1}{3}\,du = -\frac{1}{3}\cos(u) = -\frac{1}{3}\cos(3x+9)$$

$$\int x\sqrt{1-x^2}\,dx \qquad u = 1-x^2$$

$$du = -2x\,dx$$

$$-\frac{1}{2}\,du = x\,dx$$

$$\int x\sqrt{1-x^2}\,dx = \int \sqrt{1-x^2}\;x\,dx = \int \sqrt{u}\left(-\frac{1}{2}\right)du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}$$

$$= -\frac{1}{3}\left(1-x^2\right)^{3/2}$$

Check: $\frac{d}{dx} -\frac{1}{3}(1-x^2)^{3/2} = -\frac{1}{3}\,\frac{3}{2}\left((1-x^2)\right)^{1/2} \cdot (-2x)$

$$= + x\left((1-x^2)\right)^{1/2}$$

$$= x\sqrt{1-x^2} \quad \checkmark$$

---

e.g. $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$

$u = \cos(x)$
$du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$$\int \tan(x)\,dx = \int -\frac{du}{u} = -\ln(|u|)$$

$$= \ln\left(|u|^{-1}\right)$$

$$= \ln\left(|u^{-1}|\right)$$

$$= \ln\left(|\cos(x)^{-1}|\right)$$

$$= \ln\left(|\sec(x)|\right) + C$$