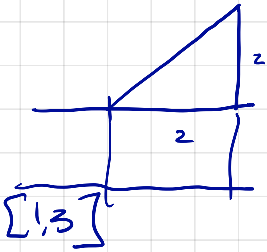
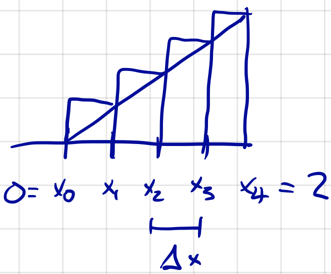
Definite Integral

Consider $f(x) = x$ on $[0, 2]$



area = 4



Approximate area:

area of first rect

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$\Delta x = \frac{1}{2}$

$x_1 = \frac{1}{2} + 1$

$x_2 = 1 + 1$

$x_3 = \frac{3}{2} + 1$

$x_4 = 2 + 1$

Approx is 2.5

$v(t) = t$ on $[1, 3]$ (v in m/s, say)

Distance traveled?

estimate on 4 time intervals

$\Delta t = \frac{1}{2}$ s.

$t_1 = 3/2$

$t_2 = 2$

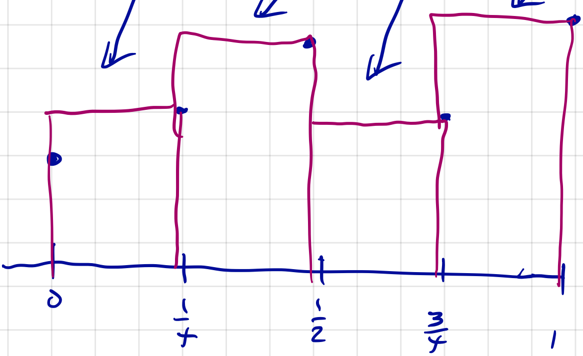
$= 5/2$

3

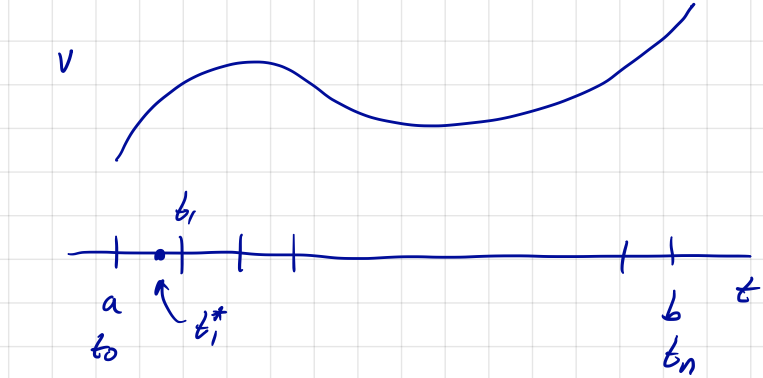
Distance problem

t	0	15	30	45	60
v	17	21	24	21	25

net distance $\frac{1}{4}\cdot 21 + \frac{1}{4}\cdot 24 + \frac{1}{4}\cdot 21 + \frac{1}{4}\cdot 20$



How could we do a better job? More sample points, shorter intervals



n intervals width $\Delta t = \dfrac{b-a}{n}$

$v(t_1^*)\Delta t \longrightarrow$ distance traveled in first time interval

Add them up

$$\sum_{k=1}^{n} v(t_k^*)\Delta t \rightsquigarrow \text{approximate net distance}$$

Now take $n \to \infty$

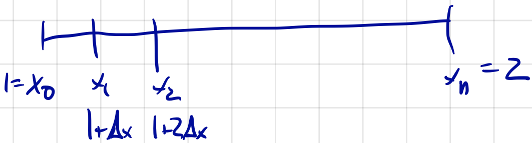
Same process works for area

General case:

$\frac{1}{2}\cdot 2\cdot 2 = 2?$

$f(x) = x$ on $[1,3]$

$\Delta x = \frac{3-1}{n}$ (subinterval length)



$x_k = 1 + k\Delta x$ $\quad$ $x_k^* = x_k$ for simplicity

Area estimate $\quad f(x_1)\Delta x + \cdots + f(x_n)\Delta x$

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} [1 + k\Delta x]\Delta x$$

Riemann Sum

$$= \sum_{k=1}^{n} \Delta x + \sum_{k=1}^{n} k \Delta x^2$$

$$= \Delta x \underbrace{\sum_{k=1}^{n} 1}_{\underbrace{1 + \cdots + 1}_{n}} + \Delta x^2 \underbrace{\sum_{k=1}^{n} k}_{1+2+\cdots+n}$$

$$\Delta x = \frac{2}{n}$$

$$= \frac{2}{n} \cdot n + \frac{4}{n^2} \left(1 + 2 + \cdots + n \right)$$

$$1 + \cdots + n$$

$$n + \cdots + 1$$

$n+1 + \cdots + n+1$ $\leftarrow$ n copies!

$$n(n+1) = 2(1 + \cdots + n)$$

$$1 + \cdots + n = \frac{n(n+1)}{2}$$

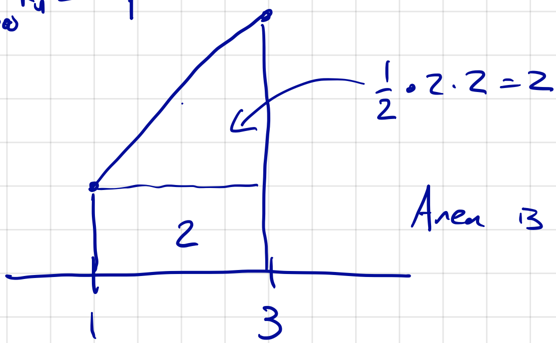
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

So $R_n = 2 + \frac{4}{n^2} \frac{n^2+1}{2}$

$= 2 + 2\left(1 + \frac{1}{n^2}\right)$

$= 4 + \frac{2}{n^2}$

$\lim_{n\to\infty} R_n = 4$



Area is 4!

This final number is denoted by

$$\int_1^3 f(x)\,dx = 4$$

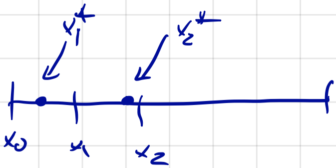
called the definite integral.

In general:

$f(x)$ on $[a,b]$

$$\Delta x = \frac{b-a}{n}$$

$$x_k = a + k\,\Delta x$$



x_k^* in $[x_{k-1}, x_k]$

$$R_n = \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n$$

if the limit exists and does not depend on the sample points.

is bounded and has finitely many disconts.

Promise: If f is continuous, or has only finitely many jump discontinuities,

$$\int_a^b f(x)\,dx \quad \text{exists}$$

Note: If $v(t)$ tells you velocity

$$\int_a^b v(t)\,dt$$ tells you net change in position

$$v(t_k)\Delta t$$

net change over interval k of position

Q: If $v(t) < 0$ everywhere is

$$\int_a^b v(t) < 0?$$

Why does this make sense?

We can interpret $\int_a^b f(x)\,dx$ as area between x-axis and f,

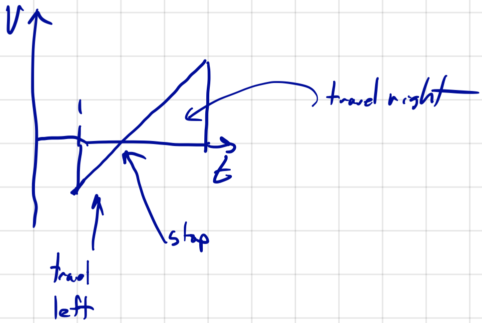
but it is <u>signed</u> area.

Why care about signed area? We don't

really. But:

velocity of an ant

e.g. $v(t) = -2 + t$ cm/s $1 \leq t \leq 4$



$v(1) = -1$ → travels left at -1 cm/s

$v(2) = 0$ stops!

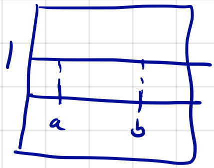
$v(3) = 1$ travels right at 1 cm/s

$$\int_1^3 v(t)\,dt = -\frac{1}{2} + \frac{1}{2} \cdot 2 \cdot 2 = -\frac{1}{2} + 2 = \frac{3}{2} \text{ cm}$$

The bug travels to the right by 1.5 cm

We need the cancellation here.

Properties



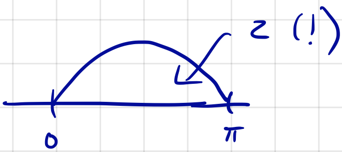
a) $\int_a^b 1\,dx = b-a$

b) $\int_0^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

c) $\int_c^b f(x)+g(x)\,dx = \int_0^b f(x)\,dx + \int_a^b g(x)\,dx$

Mix and match!

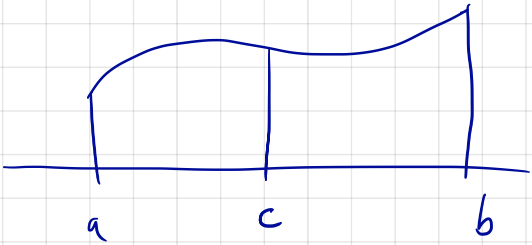
$\int_0^\pi \sin(x)\,dx = 2$



$\int_0^\pi x\,dx = \frac{\pi^2}{2}$

$$\int_0^\pi \sin(x) + 8x\,dx = 2 + 8\frac{\pi^2}{2} = 2+4\pi^2$$

d) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



e) $\int_a^a f(x)\,dx = 0$

f) $\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Algebraically: $\Delta x = \frac{b-a}{n} \to \frac{a-b}{n} = -\frac{(b-a)}{n}$.

But it's about net change.

If the ball goes up 1m from $t=0$ to $t=3$

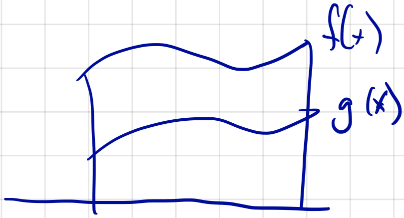
it goes down 1m from $t=3$ to $t=0$.

g) If $f(x) \geq 0$ $\int_a^b f(x)dx \geq 0$.

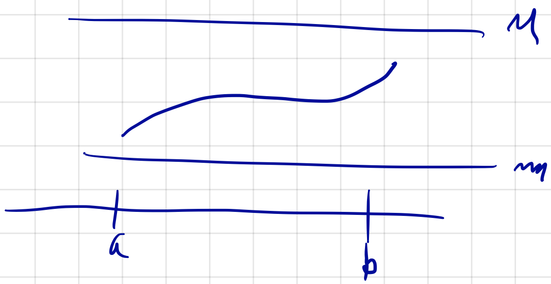
$a \leq b$

h) If $f(x) \geq g(x)$ $\int_a^b f(x) \geq \int_a^b g(x)dx$

$a \leq x \leq b$



i)



$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$