Applied Optimization

(a.k.a. What's this good for anyway?)

We're going to look at ^word problems where we want to maximize or minimize a desired quantity. (minimize cost, time. Maximize profit, speed)

Two main tools

1) Etreme Value Theorem.

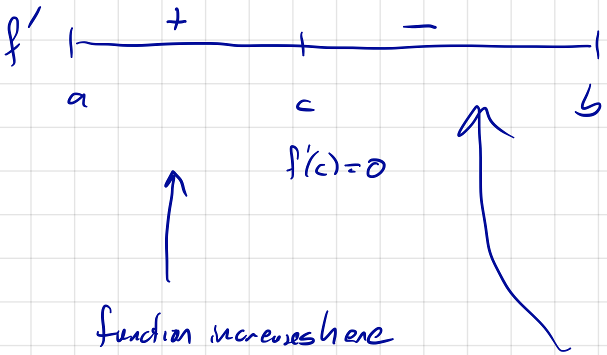
$f(x)$ on $[a,b]$.

closed, bounded.

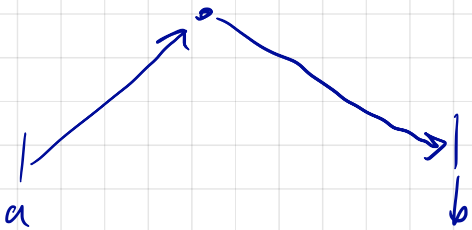
check crit pts, end points. min/max value is guaranteed to be at one of these few spots.

2) Concavity method.

If f is defined on an interval (a,b), possibly infinite, and $f'(c)=0$ and $f''(x)<0$ on (a,b), then f admits an absolute maximum at c.



($f''<0 \Rightarrow f'$ decreasing).

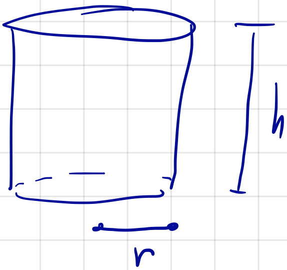


And if $f''(x)>0$ on (a,b) then f has an abs min at c.

Let's see an example.

Suppose a can has fixed volume V. What dimensions for the can minimize surface area.

1) Read problem!
2) Draw a picture; Label it.



3) Introduce "Q," the quantity to optimize, and write it in terms of other vars

$$A = 2\pi r h + 2\pi r^2$$

4) Use relations to express in terms of just 1 variable.

$$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$$

$$A = \frac{2V}{r} + 2\pi r^2$$

5) Now apply calculus.

$$\frac{dA}{dr} = -\frac{2V}{r^2} + 4\pi r \rightarrow \text{crit pts?} \quad 4\pi r = \frac{2V}{r^2}$$

$$r^3 = \frac{V}{2\pi}$$

$$\frac{d^2A}{dr^2} = \frac{4V}{r^3} + 4\pi > 0$$

$$r = \left(\frac{V}{2\pi}\right)^{1/3}$$

for $r > 0$!

So, if there is a crit pt, there is an abs min there!

$$r = \left(\frac{V}{2\pi}\right)^{1/3}$$

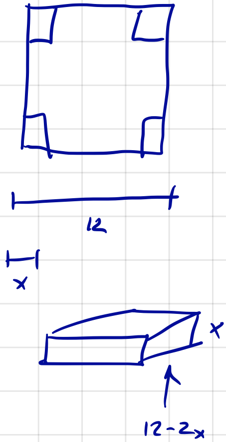
$$h = \frac{V}{\pi r^2} = \frac{V}{\pi}\left(\frac{2\pi}{V}\right)^{2/3} = 2^{2/3}\left(\frac{V}{\pi}\right)^{1/3}$$

$$= 2\left(\frac{V}{2\pi}\right)^{1/3}$$

$$= 2r$$

$h = 2r$!

e.g. We are going to construct an open top box from a $12'' \times 12''$ square of tin.



What is the maximum possible enclosed volume?

Volume: $V = (12-2x)^2 \cdot x \qquad 0 \le x \le 12$

$$\frac{dV}{dx} = (12-2x)^2 - 2(12-2x)\cdot x \cdot 2$$

$$= (12-2x)\left[12-2x-4x\right]$$

$$= (12-2x)(12-6x)$$

Closed interval method: check crit pts $(x=6,\ x=2)$

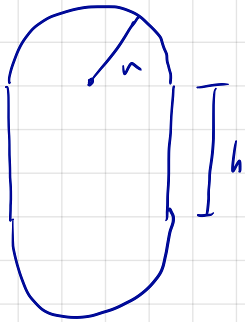
and end pts $(x=0,\ x=6)$

$V(6) = 0 = V(0)$

$V(2) = 2\cdot 8^2 = 128 \text{ in}^3$ ⟵ max volume!

e.g.: Stadium curve

If the perimeter is 440 yards, what dimensions maximize the enclosed rectangular region?



$$A = 2rh$$

$$P = 2\pi r + 2h \qquad 2h = 440 - 2\pi r$$
$$h = 220 - \pi r$$

$$A = +2r(220 - \pi r)$$

$$\frac{dA}{dr} = +440 - 4\pi r$$
$$= 440 - 4\pi r \qquad \text{one crit pt at } r = \frac{110}{\pi}$$

$$\frac{dA^2}{dr^2} = -4\pi < 0 \rightarrow \text{crit pt is an abs max.}$$

$$h = 220 - \frac{110}{\pi}\pi = 110.$$ (perimeter equally split between sides, curves!)

What about maximizing enclosed area?

$A = \pi r^2 + 2rh$ $\qquad h = 220 - \pi r$ still

$A = \pi r^2 + 2r(220 - \pi r)$

$\frac{dA}{dr} = 2\pi r + 440 - 4\pi r$

$= 440 - 2\pi r$

Again, $A'' = -2\pi < 0 \rightarrow$ one abs max.

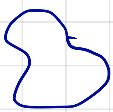
$r = \frac{220}{\pi}$ $\qquad h = 0$

$\uparrow$

(!)

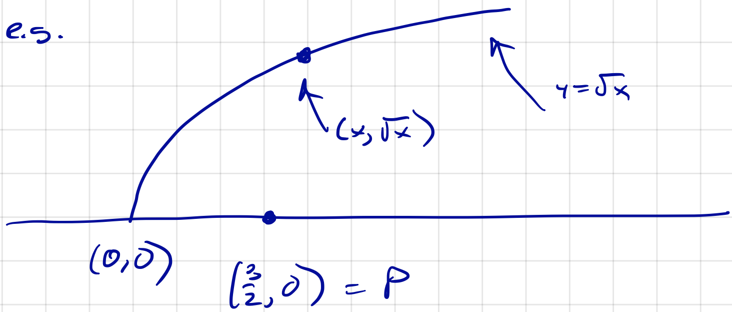
Forget the sides, just make a circle.

(Example of something called the iso-perimetric inequality:



$\rightarrow$ to maximize interior area, use a circle!)

e.g.



Find spot on curve closest to P.

distance: $\left(\left(\frac{3}{2} - x\right)^2 + (0 - \sqrt{x})^2 \right)^{1/2}$

Useful trick: minimizing distance is same as minimizing distance sq.

$+x^2 - 3x + \left(\frac{3}{2}\right)^2 + x$

$+x^2 - 2x + \left(\frac{3}{2}\right)^2$

$$D = \left(\frac{3}{2} - x\right)^2 + (\sqrt{x})^2 = \left(\frac{3}{2} - x\right)^2 + x$$

$$D' = 2\left(\frac{3}{2} - x\right) \cdot (-1) + 1$$

$$= -3 + 2x + 1$$

$$= -2 + 2x \qquad D' = 0 \Rightarrow x = 1$$

$D'' = 2 > 0 \rightarrow$ crit pt is an abs min.

Min distance: $D = \left(\left(\frac{1}{2}\right)^2 + 1^2\right)^{1/2} = \left(\frac{5}{4}\right)^{1/2} = \frac{\sqrt{5}}{2}$.