Where does exponential growth come from?

$$f(t) = C_0 \, 2^{t/a}$$

$C_0 \to f(0)$

$a \to$ time to double: $f(t+a) = 2 f(t)$ always.

Since $2 = e^{\ln(2)}$

$$2^{t/a} = \left(e^{\ln(2)}\right)^{t/a} = e^{t \ln(2)/a}$$

I.e. $f(t) = C_0 e^{kt}$ $\quad k = \dfrac{\ln(2)}{a}$ in our case.

Any exponential growth/decay function has the form

$$f(t) = C_0 e^{kt} \quad \text{for constants } C_0, t.$$

Here's the deal: $\boxed{f'(t) = k\left[C_0 e^{kt}\right] = k f(t)}$

rate of growth/decay

proportional to amount of stuff

E.g., in a petri dish of bacteria,

the growth rate (bacteria per minute) is proportional to the number of bacteria (at least until competition for food sets in).

Subtle variation: Newtons Law of Cooling.

Object: coffee, house, corpse has temp $T(t)$

Environment has temp T_0.

Newton's law of cooling states

$$T'(t) = k(T - T_0)$$

rate of change of temp is proportional to the temperature difference between the object and its surroundings.

1) $T' = 0$ if $T = T_0$ (no change if same temp)

2) $T' < 0$ if $T > T_0$ ($T - T_0 > 0$ so $k(T - T_0) < 0$ implies $k < 0$.

3) $u(t) = T(t) - T_0$ (temp with a new 0)

$$u'(t) = T'(t) = k(T - T_0) = ku.$$

$$u(t) = u_0 e^{kt}$$ u_0 some constant.

4) Since $k < 0$, $\lim_{t \to \infty} e^{kt} = 0$ and $\lim_{t \to \infty} u(t) = 0$.

Since $T(t) = u(t) + T_0$, $\lim_{t \to \infty} T(t) = 0 + T_0 = T_0$

(eventual temp is surroundings)

5) $u_0 = T(0) - T_0 = 70° - 20° = 50°C$

6) $u(t) = u_0 e^{kt} = 50 e^{kt}$

$T(10) = 60 \to u(10) = 40.$

$40 = 50 e^{10k}$ $\quad \ln\left(\frac{4}{5}\right) = 10k$

$k = \frac{1}{10} \ln\left(\frac{4}{5}\right)$ $\quad k = -0.0223$

7) $u(t) = u_0 e^{kt}$

$$T(t) = T_0 + u(t)$$

$$= T_0 + u_0 e^{kt}$$

$$= 20 + 50 e^{\ln\left(\frac{4}{5}\right) t/10}$$

$$T(20) = 20 + 50\, e^{2 \ln(4/5)}$$

$$= 20 + 50 \left(\frac{4}{5}\right)^2$$

$$= 52^\circ C$$