Consider $f(x) = x^{1/3}$

Graph is the curve $y = x^{1/3}$

$$y^3 = x$$

$y = x^3$

swap

$y^3 = x$
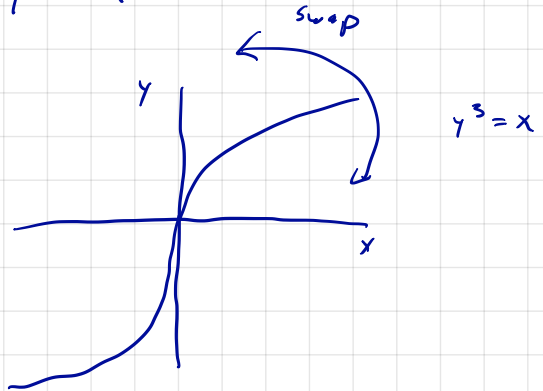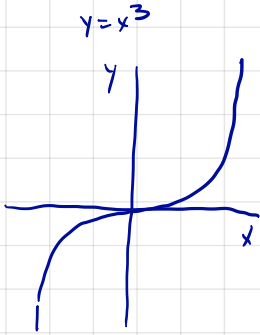
How can I compute $f'(x)$?
Alt: how can I find $\dfrac{dy}{dx}$ if $y = x^{1/3}$?

Technique is called implicit differentiation.

$$y^3 = x$$

$$\frac{d}{dx} y^3 = \frac{d}{dx} x$$

$$3y^2 \frac{dy}{dx} = 1$$

↑

This is the chain rule. $\left(\text{If you want, } \frac{dy^3}{dy} \cdot \frac{dy}{dx} \right)$

$$\frac{dy}{dx} = \frac{1}{3y^2}$$

⌐ The y is unfortunate.

I want a function of x!

But $\qquad y^3 = x$

$$y = x^{1/3}$$
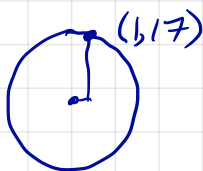$$y^2 = x^{2/3}$$

$$\frac{dy}{dx} = \frac{1}{3x^{2/3}} = \frac{1}{3} x^{-2/3}$$

Silly geometric fun:

The point $(1,4)$ lies on the curve $x^2+y^2=17$.
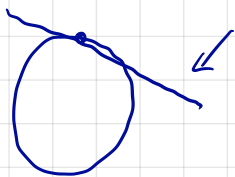
Find the slope of the tangent line at this point.

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}17$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

At $(1,4)$ The slope is then $-\frac{1}{4}$

$$(y-4) = -\frac{1}{4}(x-1)$$

$$y = 4 - \frac{1}{4}(x-1)$$

(1,17)

e.g. $xe^y = x - y$

$$\frac{d}{dx}\left(xe^y\right) = \frac{d}{dx}\left(x - y\right)$$

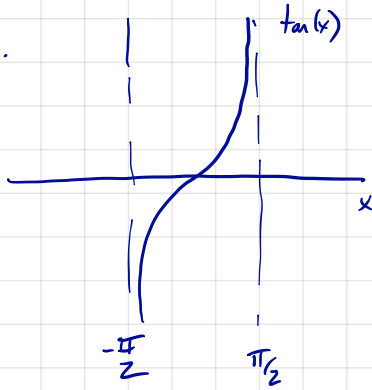$$e^y + xe^y \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$xe^y \frac{dy}{dx} + \frac{dy}{dx} = 1 - e^y$$

$$\frac{dy}{dx} = \frac{1 - e^y}{1 + xe^y}$$

One point on the line is $(x, y) = (0, 0)$

At that point, $\dfrac{dy}{dx} = \dfrac{1 - e^0}{1 + 0e^0} = \dfrac{0}{1} = 0$

e.g.

$\tan(x)$

$-\frac{\pi}{2}$ $\pi/_2$

$\arctan(x)$ is an angle y with $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and

$$\tan(y) = x$$

$$\tan(\arctan(x)) = x$$

$\arctan(\tan(y)) = y$ so long as $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \arctan(x)$

$\tan(y) = x$

$\frac{d}{dx}\tan(y) = \frac{d}{dx}x$

$\sec^2(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

Rather have x. But

$\sec^2(y) - \tan^2(y) = 1$

$\sec^2(y) = 1 + \tan^2(y) = 1 + x^2$

So $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.

That is $\dfrac{d}{dx} \arctan(x) = \dfrac{1}{1+x^2}$. See text for remaining ones.

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Now you try:

Find $\dfrac{dy}{dx}$ if $\quad y \sin(x) = x^2 - y^2$

Find $\dfrac{d^2y}{dx^2}$ if $\quad x^2 + 2y^2 = 2$