About justifying infinite limits:

Consider $f(x) = \dfrac{5}{3-x}$.

What is $\lim_{x \to 3^+} f(x)$?

top: $\lim_{x \to 3^+} 5 = 5$

$\frac{5}{0}$ looks like it might be infinite.
But what sign?

bottom $\lim_{x \to 3^+} 3-x = 0$

For x near 3, $x > 3$ $3-x < 0$.
E.g. $x = 3.01$ $3-x = -0.01$.

I'll indicate this by 0^-.

$\frac{5}{0^-} \Rightarrow -\infty$ (5 divided by a really small negative number is a large negative number)

$\frac{+}{0^+}$, $\frac{-}{0^-} \Rightarrow +\infty$

$\frac{+}{0^-}$ $\frac{-}{0^+} \Rightarrow -\infty$

$\frac{0}{0^\pm} \to$ indeterminate.

Present solutions to WS 2-2, 6, 7, 8.

Rules for working with limits.

Limits behave well with a number of common operations:

$$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} f(x) = M, \quad \text{e.g.}$$

Then $$\lim_{x \to a} \left(f(x) + g(x)\right) = L + M = \left(\lim_{x \to a} f(x)\right) + \left(\lim_{x \to a} g(x)\right).$$

$$\lim_{x \to a} f(x) - g(x) = L - M$$

$$\lim_{x \to a} f(x) g(x) = LM$$

Division is the only interesting one. Stay tuned.

Two more:

$$\lim_{x \to a} c = c \quad , \text{ any } \quad c \in \mathbb{R}$$

$$\lim_{x \to a} x = a .$$

The rules are intuitive!

$$\lim_{x \to a} x^2 - 2x + 3 = \lim_{x \to a} x^2 + \lim_{x \to a} -2x + \lim_{x \to a} 3$$

$$= \left(\lim_{x \to a} \cancel{x}\right)\left(\lim_{x \to a} x\right) + \lim_{x \to a} (-2) \lim_{x \to a} x + \lim_{x \to a} 3$$

$$= a \cdot a + (-2) \cdot a + 3$$

$$= a^2 - 2a + 3$$

i.e. just substitute $x = a$!

I'll say $f(x)$ has the Direct Substitution Property at a

if $\lim_{x \to a} f(x) = f(a)$.

From limit rules,

every polynomial has the direct substitution property at every point in its domain.

Similarly: $\lim_{x \to a} x^{\frac{1}{n}} = a^{\frac{1}{n}}$ at any point in the domain.

These limits are boring. We wouldn't need the limit concept if this was all there is to it. But its good to know the boring stuff so you can focus on the interesting stuff.

Division is subtle.

$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} g(x) = M$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \qquad \boxed{\text{so long as } M \neq 0}$$

e.g. $$\lim_{x \to 2} \frac{1-2x}{3x^2+1} = \frac{\lim_{x\to 2} 1-2x}{\lim_{x\to 2} 3x^2+1} = \frac{1-4}{3\cdot 4+1} = \frac{-3}{13}$$

$\neq 0$ ok!

Direct substitution works for rational functions!

If $L \neq 0$ and $M = 0$, often the one-sided limits are $\pm\infty$. You need to do a sign analysis.

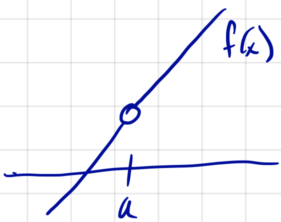
$$\frac{5}{0^+} \Rightarrow +\infty, \text{ etc.}$$

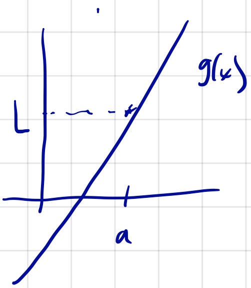
For $\frac{0}{0}$, more work! (This is often where all the fun is)

Key tools:

① $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L$ & $\lim_{x \to a^+} f(x) = L$.

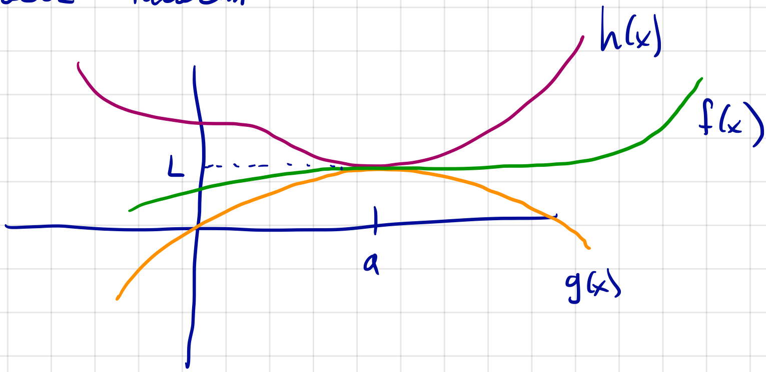
② "Limits don't care about one point."
If $f(x) = g(x)$ except at $x = a$, and if $\lim_{x \to a} g(x) = L$,





Then $\lim_{x \to a} f(x) = L$.

③ Squeeze theorem



if $\quad g(x) \leq f(x) \leq h(x)$

and $\quad \lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)$ then

$$\lim_{x \to a} f(x) = L \text{ also.}$$