Last class: On worksheet derived average rate of change formula. For distance:

$$\frac{\text{change in distance}}{\text{change in time}}$$

Over time interval $[t_0, t_1]$, with $d(t)$ distance traveled

$$\frac{d(t_1) - d(t_0)}{t_1 - t_0} \qquad \frac{\text{change in dist}}{\text{change in t}}$$

or, over time interval $[t_0, t_0+h]$

$$\frac{d(t_0+h) - d(t_0)}{h}$$

h is the length of the time interval. So $h=0$ should get you speed right at t_0! But no:

if $h=0$: $\quad \dfrac{d(t_0) - d(t_0)}{0} = \dfrac{0}{0}$ ← wow. big uh oh.

uh oh! → 0

Instead, we can approximate the speed at $t = t_0$ by taking h very small, with the hope that as h goes to 0 the approximation gets better and better.

12. Instead, we can work with average speeds over short time intervals near time $t = 41$ minutes. Use the spreadsheet to compute the average speeds over the time intervals $[41, 41 + h]$ for:

(a) $h = 1$ minutes 1.113414664 miles/minute

(b) $h = 0.1$ minutes 1.108981788 miles/minute

(c) $h = 0.01$ minutes 1.108373364 miles/minute

(d) $h = 0.001$ minutes 1.108310877 miles/minute

(e) $h = 0.0001$ minutes 1.108304617 miles/minute

(f) $h = 0.00001$ minutes 1.108303986 miles/minute

Looks like we're settling in around

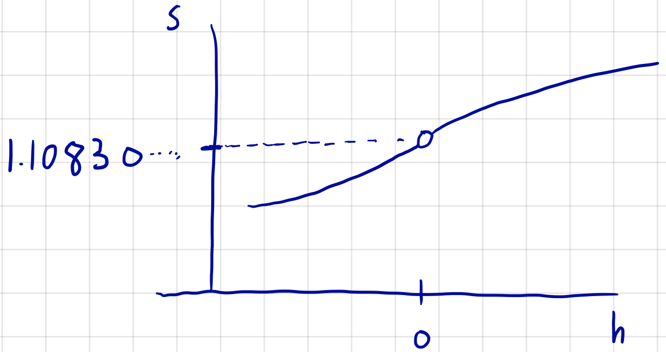
1.10830 miles/minute

error $\approx 10^{-5} \frac{\text{miles}}{\text{minute}} = 38$ inches/hour.

($\Delta t = \frac{1}{2}$ milisecond!)

$$s(h) = \frac{d(41+h) - d(41)}{h}$$

is perfectly well defined near $h=0$, but not at $h=0$.



Graph of s has a hole at $h=0$.

We need to cope with functions with holes and to discuss the values they are supposed to have to "fill in" the hole.

More examples...

Average rates of change aren't just for speed!

If a quantity depends on time, we compute average rates of change this way:

$$\frac{\text{change in quantity}}{\text{change in time}} \longrightarrow \underline{\text{average}} \text{ rate of change}$$

e.g. our friends the caribou:

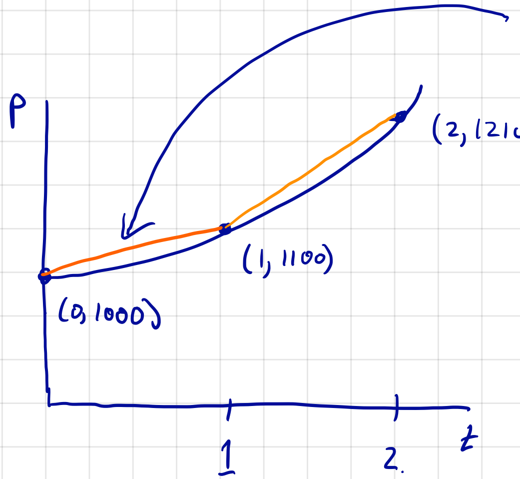
$$p(t) = 1000\,(1.1)^t$$

Compute the average rate of change in the population over the first year and over the 2nd year:

first year: $\dfrac{p(1)-p(0)}{1-0} = \dfrac{1100-1000}{1} = 100 \ \dfrac{\text{caribou}}{\text{year}}$

second year $\dfrac{p(2)-p(1)}{2-1} = \dfrac{1210-1100}{1} = 110 \ \dfrac{\text{caribou}}{\text{year}}$

secant line: two ends on graph!



$$\frac{\Delta p}{\Delta t}$$

Connection w/ geometry: slope of secant line $\dfrac{1100 - 1000}{1 - 0} = 100 \ \dfrac{\text{caribou}}{\text{year}}$

The slope of the secant line on the graph tells you an average rate of change.

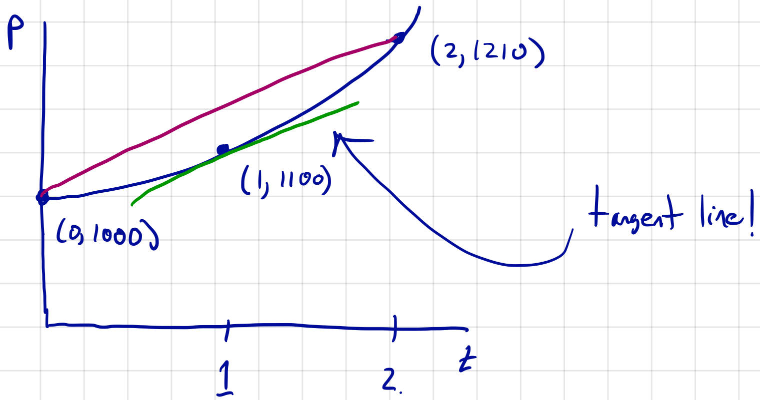
What about <u>right</u> <u>at</u> $t = 1$?

100 is probably too little
110 is probably too much.

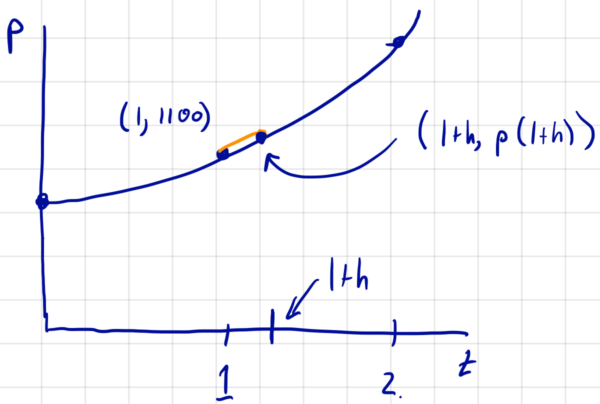
One estimate: $\dfrac{100 + 110}{2} = 105 \ \dfrac{\text{caribou}}{\text{year}}$

that's a slope, too! $\dfrac{p(2) - p(0)}{2 - 0} = \dfrac{1210 - 1000}{2}$

$= \dfrac{210}{2} = 105$



For simplicity:



Average rate of change over interval
(Slope of secant line over interval) $\to [1, 1+h]$

is $$\frac{p(1+h)-p(1)}{h} = \frac{1000(1.1)^{1+h} - 1000(1.1)}{h}$$
$$= 1100\left[\frac{(1.1)^h - 1}{h}\right]$$

For small choices of h, get an average rate of change over a short interval

e.g. $h = \frac{1}{2}$ (1/2 year)

avg. rate of change 107.38 caribou per year.

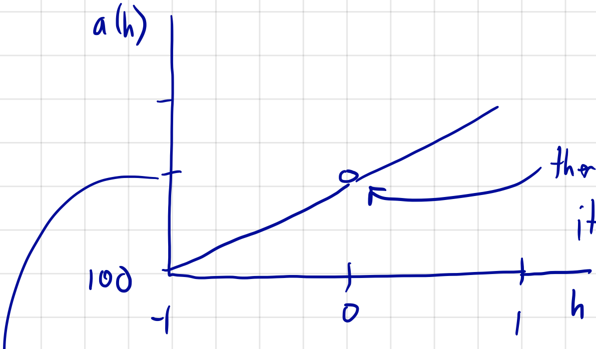
$h = 0.1$	105.34...
$h = 0.01$	104.89...
$h = 0.0001$	104.84169
$h = 0.0000001$	$\underbrace{104.841}$99

↳ look like they are settling in.

But $h = 0$ is a no no: $\frac{(1.1)^0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$

$$a(h) = 1000\,\frac{\left((1.1)^h - 1\right)}{h} \rightarrow \text{average rate of change over}$$

$[1,\ 1+h]$ ($h<0$ is ok!)



there's a hole in the graph!
it isn't defined here!

value is $\approx$ 104.8411

we need to be able to talk about the value there. it is

a) the (instantaneous) rate of change

b) the slope of the tangent line to the graph.

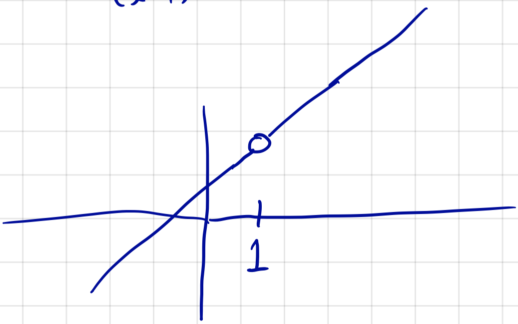
a) is super important

b) is less so, but becomes super important because of a)

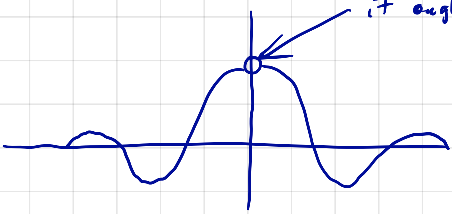
Another function with a hole:

$$\frac{x^2-1}{x-1} \qquad x=1: \quad \frac{0}{0} \leftarrow \text{uh oh!}$$

$$\frac{(x-1)(x+1)}{(x-1)} = x+1 \quad \underline{\text{except}} \text{ at } x=1.$$



Another: $\dfrac{\sin(x)}{x}$ at $x=0$: $\dfrac{0}{0}$



it ought to have a value,
but it can't: division by 0!

$$\frac{\sin(0.01)}{0.01} \approx 0.99998$$

$$\frac{\sin(0.001)}{0.001} \approx 0.99999983$$

$\to 1$?

$\frac{0}{0}$ often, but not always, signals a function with a hole.

To deal with the hole we introduce a new concept.

$$\boxed{\text{We say } \lim_{x \to a} f(x) = L \text{ if the values of } f(x) \text{ can be made as close to } L \text{ as we please by taking } x \text{ close enough to } a \text{ (but maybe not } a \text{ itself).}}$$

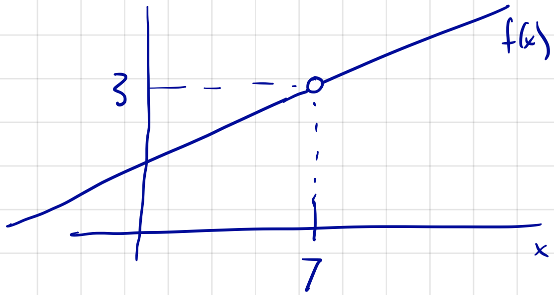
$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{h \to 0} 1000 \left(\frac{(1.1)^h - 1}{h} \right) = \underbrace{1100 \ln(1.1)}, \approx 104.81197784757$$

↳ how do I know this! (teaser)

Picture:



$$\lim_{x \to 7} f(x) = 3$$