MATH 253
Quiz 9

Name: SOLUTIONS
Due at 11:59 pm on April 12.

**Instructions:** (10 points total – 5 pts each) Show all work for credit. You may use your book, but no other resource. **GS:** Scan ~~THREE~~ FOUR pages for your solutions.

1. Consider the two dimensional vector field

$$\mathbf{F}(x, y) = \left\langle e^{xy}(y\sin(x) + \cos(x)),\ xe^{xy}\sin(x) + \frac{1}{y} \right\rangle$$

defined on all ~~of $\mathbb{R}^2$~~. the upper half-plane in $\mathbb{R}^2$,

(a) Prove that **F** is conservative, then find its potential function $f(x, y)$.

With $P = e^{xy}(y\sin x + \cos x)$, $\frac{\partial P}{\partial y} = e^{xy}[\sin x] + xe^{xy}y\sin x + xe^{xy}\cos x$

$= e^{xy}(\sin x + xy\sin x + x\cos x)$

$Q = xe^{xy}\sin x + \frac{1}{y}$, $\frac{\partial Q}{\partial x} = [xe^{xy}]\cos x + [xe^{xy}y + (1)e^{xy}]\sin x$

$= e^{xy}(x\cos x + xy\sin x + \sin x)$

Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on the upper-half plane (simply-connected, open, etc.)

therefore, $\vec{F}$ is conservative.

Moreover, if $Q = \frac{\partial f}{\partial y} = xe^{xy}\sin x + \frac{1}{y}$, then $\int \frac{\partial f}{\partial y}\,dy = \int xe^{xy}\sin x + \frac{1}{y}\,dy$

$= e^{xy}\sin x + \ln(y) + c(x) = f(x,y)$ where $c(x)$ is a function of $x$ alone.

Then $\frac{\partial f}{\partial x} = e^{xy}(y\sin x + \cos x) + \ln(y) + c'(x) = P \Rightarrow c(x) = C$. Thus

(b) Letting $C$ be the line segment joining $(0, 1)$ to the point $(0, \frac{\pi}{2})$, compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$\boxed{f(x,y) = e^{xy}\sin x + \ln|y| + C}$

For (b) use $f(x,y)$ the potential funct...

$$\int_C \vec{F}\cdot d\vec{r} = f(0,\pi/2) - f(0,1) = \left(e^{0(\pi/2)}\sin 0 + \ln(\pi/2)\right) - \left(e^0\sin(0) - \ln(1)\right)$$

$$= (0 + \ln(\pi/2)) - (0)$$

$$= \boxed{\ln(\pi/2)}$$

2. In this problem you will show that the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$$

for the vector field $\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and $C$ any positively oriented simple closed circle enclosing the origin. Note that the vector field $\mathbf{F}$ is not defined at the origin, so the domain is the punctured plane.

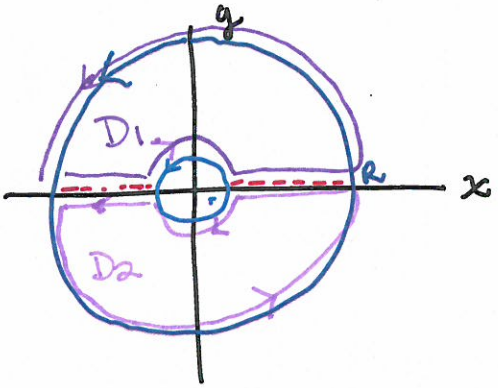
(a) Letting $C_r$ and $C_R$ denote the circles of radius $r < R$, first compute by parameterizing the circle that $\oint_{C_R} \mathbf{F} \cdot d\mathbf{r} = 2\pi$.

$\vec{r}(t) = \langle R\cos t, R\sin t \rangle \quad 0 \le t \le 2\pi \qquad \vec{r}\,'(t) = \langle -R\sin t, R\cos t \rangle \qquad$ Note: $x^2+y^2=R^2$.

$$\vec{F}(\vec{r}(t)) = \left\langle -\frac{R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle = \frac{1}{R}\langle -\sin t, \cos t \rangle.$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle -\frac{1}{R}\sin t, \frac{1}{R}\cos t \right\rangle \cdot \langle -R\sin t, R\cos t \rangle \, dt = \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \boxed{2\pi}$$

(b) Now use the extended Green's theorem to compute that $\oint_{C_r} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_R} \mathbf{F} \cdot d\mathbf{r}$. See picture.



$C_r$ = inner circle

$C_R$ = outer circle

Red line segments help with extension

Caution: Watch orientation of $C_r$, $C_R$ in your computations.

Bust the annulus into two semi-annular regions so that Green's Theorem can be applied. $D_2$ = bottom annulus $D_1$ = top. I will use $A$ for the annular region

$$\oint_{\partial D_1} \vec{F}\cdot d\vec{r} + \oint_{\partial D_2} \vec{F}\cdot d\vec{r} = \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA + \iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$ by Green's Theorem.

$$= \oint_{C_R} \vec{F}\cdot d\vec{r} - \oint_{C_r} \vec{F}\cdot d\vec{r} = I$$

(continued on next page ...)

Notice: In applying Green's Theorem, $C_r$ is traversed in the negative orientation.

Computing:

- $P = -y(x^2+y^2)^{-1}$, then

$$\frac{\partial P}{\partial y} = -y(-1)(x^2+y^2)^{-2}(2y) - (x^2+y^2)^{-1}$$

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{(x^2+y^2)}$$

$$= \frac{2y^2-(x^2+y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

- $Q = x(x^2+y^2)^{-1}$, then $\frac{\partial Q}{\partial x} = x(-1)(x^2+y^2)^{-2}(2x) + (1)(x^2+y^2)^{-1}$

$$= (x^2+y^2)^{-2}\left(-2x^2 + (x^2+y^2)\right)$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

<u>and</u> (miraculously) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

Thus, $I = 0$! and since $I = \oint_{C_R} \vec{F}\cdot d\vec{r} - \oint_{C_r} \vec{F}\cdot d\vec{r} = 0$

$$\oint_{C_R} \vec{F}\cdot d\vec{r} = 2\pi = \oint_{C_r} \vec{F}\cdot d\vec{r}.$$

(c) Green's Theorem requires an open simply-connected domain and here there is a "hole" or "puncture" at (0,0) since $\vec{F}$ is <u>not</u> defined there.