Name:

Stokes's Theorem: If C is the boundary of a 'nice' region $\mathcal{S}$,

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot d\mathbf{r}$$

so long as the normal $\mathbf{n}$ and the orientation of C are compatible.

In the two problems below you will set up, but **not evaluate** the integrals on both side of this equation where $\mathcal{S}$ is the hemisphere $x^2 + y^2 + z^2 = 1$ with $y \geq 0$ and where

$$\mathbf{F} = \langle y, z, -x \rangle .$$

The hemisphere is given the orientation with unit normal pointing **towards the origin**.

1. Write down an integral expressing $\int_C \mathbf{F} \cdot d\mathbf{r}$. Your answer should be of the form $\int_a^b g(t)\, dt$ where a and b are numbers and where $g(t)$ is an explicit function. Please do not compute the integral. Please be careful about orientation/sign.

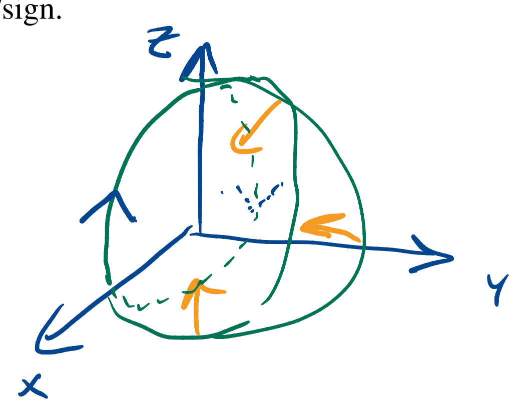
$\vec{r}(t) = \langle \cos t, 0, \sin t \rangle$

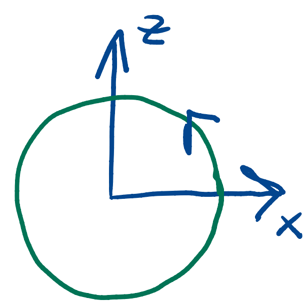
$\vec{r}\,'(t) = \langle -\sin t, 0, \cos t \rangle$

$\vec{F}(\vec{r}(t)) = \langle 0, \sin(t), -\cos(t) \rangle$

$F(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\cos^2(t)$

$$\int_C F \cdot d\vec{r} = \boxed{\int_0^{2\pi} -\cos^2(t)\, dt} = -\pi$$





Recall: $\mathcal{S}$ is the hemisphere $x^2 + y^2 + z^2 = 1$ with $y \geq 0$ and unit normal pointing towards the origin and

$$\mathbf{F} = \langle y, z, -x \rangle .$$

2. Write down an integral expressing $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \; dS$. Your answer should be in the form of an iterated integral of an explicit integrand that is a function of two parameter variables. Please do **not** compute the integral.

$$y = \sqrt{1 - x^2 - z^2}$$

$$\vec{r}(u,v) = \langle u, \sqrt{1-u^2-v^2}, v \rangle$$

$$\vec{r}_u = \left\langle 1, \frac{-u}{\sqrt{1-u^2-v^2}}, 0 \right\rangle$$

$$\vec{r}_v = \left\langle 0, \frac{-v}{\sqrt{1-u^2-v^2}}, 1 \right\rangle$$

$$\vec{r}_u \times \vec{r}_v = \left\langle \frac{-u}{\sqrt{1-u^2-v^2}}, -1, \frac{-v}{\sqrt{1-u^2-v^2}} \right\rangle$$

$$\vec{F}(\vec{r}(u,v)) = \langle \sqrt{1-u^2-v^2}, v, -u \rangle$$

$$\vec{\nabla} \times \vec{F} = \langle -1, 1, -1 \rangle$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{r}_u \times \vec{r}_v = -1 + \frac{v+u}{\sqrt{1-u^2-v^2}}$$

integrates to 0

$$\boxed{\int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} -1 + \frac{v+u}{\sqrt{1-u^2-v^2}} \, dv\, du} = \int_0^1 \int_0^{2\pi} \left[-1 + \frac{r\sin\theta + r\cos\theta}{\sqrt{1-r^2}} \right] r \, d\theta \, dr$$

$$= 2\pi \int_0^1 -r = -\pi \checkmark$$

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