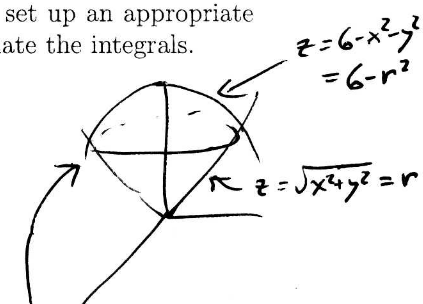
Exam 2
Math 202 F01

Name: Solutions
11/12/12

1. (12 pts.) A 3-d object is bounded below by $z = \sqrt{x^2+y^2}$ and above by $z = 6-x^2-y^2$. Its mass density is given by $\rho(x,y,z) = x^2+y^2+z^2$. Using CYLINDRICAL COORDINATES, set up an appropriate integral expression for $\bar{z}$, the z-coordinate of its center of mass. DO NOT evaluate the integrals.



circle of intersection

$z=r$ + $z=6-r^2$

$r=6-r^2$

$r^2+r-6=0$

$(r+3)(r-2)=0$

$r=-3, 2$

$$\bar{z} = \frac{\iiint_Q z\rho\, dV}{\iiint_Q \rho\, dV}$$

$$\rho = x^2+y^2+z^2 = r^2+z^2$$

$$= \frac{\int_0^{2\pi}\int_0^2\int_r^{6-r^2} z(r^2+z^2)\, r\, dz\, dr\, d\theta}{\int_0^{2\pi}\int_0^2\int_r^{6-r^2} (r^2+z^2)\, r\, dz\, dr\, d\theta}$$

2. (12 pts.) Consider the function $f(x,y) = x^3+y^3-3xy+12$.

(a) (4 pts.) Show that $(0,0)$ and $(1,1)$ are critical points of f. (They are actually the only critical points of f, but you need not show that.)

$\nabla f = \langle 3x^2-3y, 3y^2-x^2 \rangle$

so $\nabla f(0,0) = \langle 0,0 \rangle$

$\nabla f(1,1) = \langle 0,0 \rangle$

(b) (8 pts.) Apply the 2nd derivative test at each of these points, and state your conclusions from it.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$$

$D(0,0) = -9 < 0 \Rightarrow$ $(0,0)$ is a saddle

$D(1,1) = 36-9 = 27 > 0$, $f_{xx}(1,1) = 6 > 0$ $\Rightarrow$ $(1,1)$ is a local minimum

3. (7 pts.) Give an equation for the tangent plane to the surface $\underbrace{x^2 - 2y^2 + z^2 + yz}_{f(x,y,z)} = 2$ at the point $(2, 1, -1)$.

$\nabla f = \langle 2x, -4y+z, 2z+y \rangle$

$\nabla f(2,1,-1) = \langle 4, -5, -1 \rangle$

$$4(x-2) - 5(y-1) - 1(z - -1) = 0$$

$$4x - 5y - z = 4$$

4. (15 pts.–5 pts. each) The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2},$$

where T is measured in $°C$, and x, y, z are measured in meters.

(a) Find the rate of change of the temperature at the point $(2, -1, 2)$ in the direction towards the point $(3, -3, 3)$. GIVE UNITS.

$\vec{v} = (3,-3,3) - (2,-1,2) = \langle 1, -2, 1 \rangle$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle$

$\nabla T(2,-1,2) = 200e^{-x^2-3y^2-9z^2} \langle -2x, -6y, -18z \rangle \Big|_{(2,-1,2)} = 200e^{-43} \langle -4, 6, -36 \rangle$

So $D_{\vec{u}}T = \nabla T \cdot \vec{u} = 200e^{-43} \frac{1}{\sqrt{6}} \left((-4)(1) + 6(-2) + -36(1) \right) = \frac{200e^{-43}(-52)}{\sqrt{6}}$

$= -\frac{5200\sqrt{2}e^{-43}}{\sqrt{3}}$ $°C/m$

(b) At $(2, -1, 2)$, in what direction does T increase most rapidly?

Same direction as $\nabla T(2,1,-2)$

So $\langle -4, 6, -36 \rangle$ or $\langle -2, 3, -18 \rangle$

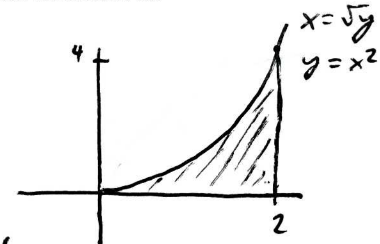
(c) What is the maximum rate of change of T at $(2, -1, 2)$, among all directions?

$\|\nabla T(2,1,-2)\| = 200e^{-43} \|\langle -4, 6, -36 \rangle\| = 200e^{-43}\sqrt{4^2+6^2+36^2}$

$= 200e^{-43}(2)\sqrt{1+3^2+18^2} = 400e^{-43}\sqrt{337}$

5. (12 pts.) Reverse the order of integration in the following integral, and then evaluate it.

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy$$



$$= \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx = \int_0^2 \left(\sqrt{x^3+1}\, y\right)\Big|_{y=0}^{x^2} dx$$

$$= \int_0^2 x^2\sqrt{x^3+1}\, dx = \frac{1}{3}\int_1^9 u^{1/2} du = \frac{1}{3}\left(\frac{2}{3}\right) u^{3/2}\Big|_{u=1}^{9} = \frac{2}{9}\left(9^{3/2} - 1^{3/2}\right)$$

$u = x^3+1$

$du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

$$= \frac{2}{9}(27-1) = \frac{52}{9}$$

6. (10 pts.) Ohm's law states that in an electrical circuit the current, I, depends on the voltage, V, and resistance, R, by

$$I = V/R.$$

Suppose at some moment $R = 100$ ohms, $V = 32$ volts, $dR/dt = 0.03$ ohms/s, and $dV/dt = -0.01$ volts/s. Determine dI/dt at that moment. GIVE UNITS. (Hint: Use the multivariable chain rule. The unit 'volt/ohm' is also called an 'ampere'.)

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt}$$

$$= \frac{1}{R}\frac{dV}{dt} + \left(\frac{-V}{R^2}\right)\frac{dR}{dt}$$

$$= \left(\frac{1}{100 \text{ ohms}}\right)\left(-.01 \frac{\text{volts}}{s}\right) + \left(\frac{-32 \text{ volts}}{100^2 \text{ ohms}^2}\right)\left(.03 \frac{\text{ohms}}{s}\right)$$

$$= -.0001 \frac{\text{volts}}{\text{ohm}\cdot s} + -\frac{.96}{10^4} \frac{\text{volts}}{\text{ohm}\cdot \text{sec}}$$

$$= -.000196 \frac{\text{amperes}}{s}$$

7. (12 pts.) Use the method of Lagrange multipliers to find the point on the sphere $\underbrace{x^2+y^2+z^2}_{g(x,y,z)} = 70$ that minimizes $f(x, y, z) = 2x + 6y + 10z$.

$\nabla f = \lambda \nabla g \Rightarrow \langle 2,6,10\rangle = \lambda\langle 2x, 2y, 2z\rangle$

So $\left.\begin{array}{l} 2 = 2x\lambda \\ 6 = 2y\lambda \\ 10 = 2z\lambda \\ \text{\& } x^2+y^2+z^2=70 \end{array}\right\}$ system of equations to solve.

Thus $x = \frac{1}{\lambda}$, $y = \frac{3}{\lambda}$, $z = \frac{5}{\lambda}$

and $\left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 = 70$

$\frac{35}{\lambda^2} = 70$

$\lambda^2 = \frac{1}{2}$

so $\lambda = \pm\frac{1}{\sqrt{2}}$

so $(x,y,z) = \pm(\sqrt{2}, 3\sqrt{2}, 5\sqrt{2})$

The value of f is smaller at the negative version, so

$(x,y,z) = (-\sqrt{2}, -3\sqrt{2}, -5\sqrt{2})$

8. (12 pts.–3 pts. each) Complete the following.

(a) The average value of a function $f(x, y)$ over a 2-dimensional region R is given by the formula:

$$\frac{\iint_R f(x,y)\,dA}{\iint_R dA}$$

(b) In spherical coordinates, dV is:

$\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 - xy^2}{x^2 + y^2}$ does not exist since: if $(x,y)\to(0,0)$ along the x-axis, so $y=0$

then $\frac{2x^2-xy^2}{x^2+y^2} = \frac{2x^2}{x^2} = 2 \to 2$

& if $(x,y)\to(0,0)$ along the y-axis, so $x=0$, then $\frac{2x^2-xy^2}{x^2+y^2} = \frac{0}{y^2} = 0 \to 0$

Since $2 \neq 0$, there is no common value that the function approaches as $(x,y)\to(0,0)$.

(d) The geometric relationship between the level curves of a function $z = f(x, y)$ and the gradient vectors $\nabla f(x, y)$ is:

$\nabla f(x,y)$ is $\perp$ to the level curve of f through (x,y). Moreover, $\nabla f(x,y)$ points toward higher level curves (i.e. $f(x,y)=c$ where c is larger).

9. (8 pts.–4 pts. each) Suppose $x = u^2 + v$, $y = u - v^2$ represents a change of coordinates for re-expressing a double integral in x, y in terms of u, v.

(a) Compute $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 1 \\ 1 & -2v \end{vmatrix} = -4uv - 1$$

(b) Give a sentence or two of informal explanation of why $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ should appear in the integral in terms of u, v. What do the parts of this expression represent geometrically? (DO NOT give a mathematical derivation of the expression.)

$dx\,dy = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ shows that the factor $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$ accounts for the way areas change when we pass between u-v coordinates and x-y ones. The bits of u-v area du dv may not be the same size as the bits of x-y area dx dy, since the coordinate change may either shrink or stretch them.