MATH202X-F01/UX1
Spring 2015

Midterm Exam 2

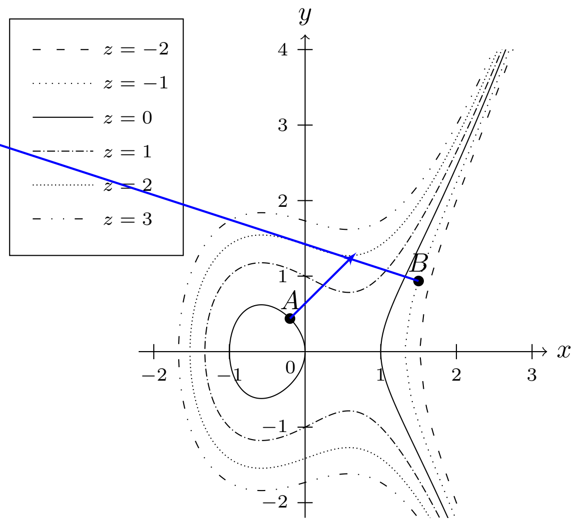
Name: Answer Key

Instructions. (0 points) You have 60 minutes. No calculators allowed. *Show all your work* in order to receive full credit.

1. Explain why $\lim_{(x,y)\to(1,0)} \frac{xy^2}{(x-1)^2+y^2}$ does not exist.

Solution: Setting $x = 1$ and letting $y \to 0$ to approach $(1, 0)$ along the line $(1, y)$, we see $\lim_{y\to 0} \frac{y^2}{y^2} = 1$. Setting $y = 0$ and letting $x \to 1$ to approach $(1, 0)$ along the line $(x, 0)$, we see $\lim_{x\to 1} \frac{0}{(x-1)^2} = 0$. Since these limits are different, $\boxed{\text{the original multivariable limit does not exist.}}$

2. The plot below shows several level curves of a function $z = f(x, y)$. At the points A and B sketch vectors representing the correct directions for ∇f. Would ∇f be longer at A or at B?



Solution:

The gradient is orthogonal to the level curves towards increasing values of z. It will be $\boxed{\text{longer at } B}$ because for the same positive change of z-value, the level curves are much more closely spaced at B than at A.

3. The pressure P (in kilopascals) of one mole of an ideal gas is determined by its temperature T (in kelvins) and volume V (in liters) according to

$$P = 8.3\frac{T}{V}.$$

If $T = 300$ kelvins, $dT/dt = 0.2$ kelvins/sec, $V = 10$ liters, $dV/dt = 0.1$ liters/sec, at what rate will the pressure be changing?

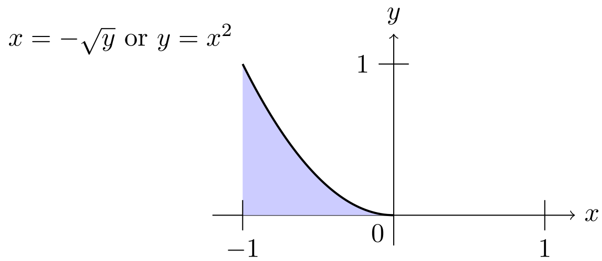
Solution: By the chain rule,

$$\begin{aligned}
\frac{dP}{dt} &= \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} \\
&= 8.3\frac{1}{V}\frac{dT}{dt} - 8.3\frac{T}{V^2}\frac{dV}{dt} \\
&= 8.3\frac{1}{10}(0.2) - 8.3\frac{300}{10^2}(0.1) \\
&= 8.3(.02 - .3) = 8.3(-.28) = \boxed{-2.324 \text{ kilopascals/sec.}}
\end{aligned}$$

4. Compute the iterated integral by switching the order of integration.

$$I = \int_0^1 \int_{-1}^{-\sqrt{y}} e^{x^3}\, dx\, dy.$$

Solution: Sketching the region of integration, we have:



So switching the order,

$$I = \int_{-1}^{0} \int_0^{x^2} e^{x^3}\, dy\, dx = \int_{-1}^{0} \left[y e^{x^3} \right]_0^{x^2} dx = \int_{-1}^{0} x^2 e^{x^3}\, dx$$
$$= \left[\frac{1}{3} e^{x^3} \right]_{-1}^{0} = \frac{1}{3}\left(1 - \frac{1}{e}\right) = \boxed{\frac{e-1}{3e}}.$$

5. Using the method of Lagrange multipliers, find the points on the circle $x^2 + y^2 = 1$ where the maxima and minima of the function

$$f(x, y) = x^2 + y$$

occur. For each of the points, indicate whether a maximum or a minimum occurs.

Solution: With $g(x, y) = x^2 + y^2$, we set $\nabla f = \lambda \nabla g$ to find $\langle 2x, 1\rangle = \lambda \langle 2x, 2y\rangle$. From $2x = \lambda 2x$ we have $2x(1 - \lambda) = 0$ so $x = 0$ or $\lambda = 1$.
If $x = 0$, from $x^2 + y^2 = 1$ we see $y = \pm 1$.
If $\lambda = 1$, from $1 = \lambda 2y$ we see $y = 1/2$, and then $x^2 + y^2 = 1$ shows $x = \pm\sqrt{3}/2$.
Evaluating f at the four points $(0, \pm 1)$, $(\pm\sqrt{3}/2, 1/2)$ shows

- $\boxed{\text{a maximum occurs at the two points } (\pm\sqrt{3}/2, 1/2)}$;
- and $\boxed{\text{a minimum at } (0, -1)}$;
- at $(0, 1)$, there is neither a maximum or minimum.

6. Find an equation of the tangent plane to the surface

$$y\cos(z+x)+z^2=5$$

at the point $(x_0, y_0, z_0) = (2, 0, -2)$.
Solution: For $f(x, y, z) = y\cos(z + x) + z^2$, we find

$$\nabla f = \langle -y\sin(z+x), \cos(z+x), -y\sin(z+x)+2z\rangle,$$

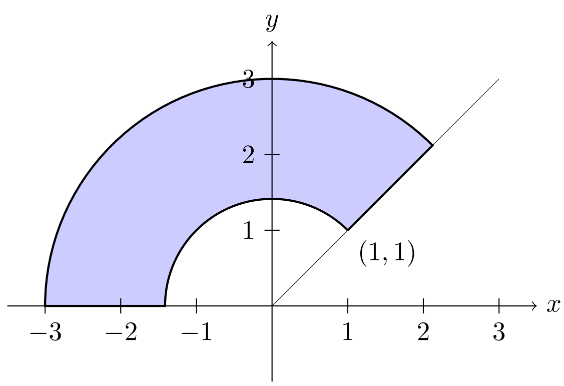
so $\nabla f(2, 0, -2) = \langle 0, 1, -4\rangle$. The tangent plane is thus given by

$$0(x-2)+1(y-0)-4(z+2)=0,$$

or

$$\boxed{y-4z=8.}$$

7. Compute $\iint_R f(x, y)\, dA$ for $f(x, y) = \sqrt{9 - x^2 - y^2}$ and R the bounded region shaded below.



Solution: Let's use polar coordinates: $f(r\cos\theta, r\sin\theta) = \sqrt{9-r^2}$ and R will have constant bounds in (r, θ). The inner circle has $(1, 1)$ on it so $1^2+1^2 = 2$ so that's $r = \sqrt{2}$ whereas the outer circle has radius 3. The boundary line $y = x$ corresponds to $\theta = \frac{\pi}{4}$ and the negative x-axis to $\theta = \pi$. Hence

$$\iint_R f(x,y)\, dA = \int_{\sqrt{2}}^{3}\int_{\frac{\pi}{4}}^{\pi} \sqrt{9-r^2}\, r\, d\theta\, dr = \int_{\sqrt{2}}^{3}\left[\theta r\sqrt{9-r^2}\right]_{\frac{\pi}{4}}^{\pi} dr = \frac{3\pi}{4}\int_{\sqrt{2}}^{3} r\sqrt{9-r^2}\, dr$$

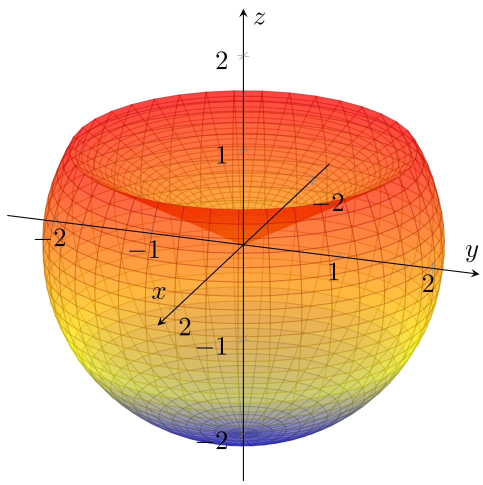
$$= \frac{3\pi}{4}\left[\left(-\frac{1}{2}\right)\frac{2}{3}\left(9-r^2\right)^{\frac{3}{2}}\right]_{\sqrt{2}}^{3} = \frac{\pi}{4}\left(0+7\sqrt{7}\right) = \boxed{\frac{7\pi\sqrt{7}}{4}.}$$

8. The mass of a solid Q is given by:

$$m = \int_0^2\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi} \rho^4\cos^2\phi\sin\phi\, d\phi\, d\theta\, d\rho.$$

(a) Describe and sketch the solid Q.

Solution:



This is a sphere of radius 2 with the inside of the half cone $0 \le \phi \le \frac{\pi}{3}$ taken out.

(b) Deduce from the equation above the density function:

Solution: $f(x,y,z) = \boxed{z^2}$

(c) Evaluate the mass m.

Solution:

$$\begin{aligned} m &= \int_0^2 \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\pi} \rho^4 \cos^2\phi \sin\phi \, d\phi \, d\theta \, d\rho = \int_0^2 \rho^4 \int_0^{2\pi} \left[-\frac{1}{3}\cos^3\phi \right]_{\frac{\pi}{3}}^{\pi} d\theta \, d\rho \\ &= \int_0^2 \rho^4 \int_0^{2\pi} -\frac{1}{3}\left((-1)^3 - \left(\frac{1}{2}\right)^3 \right) d\theta \, d\rho = \frac{3}{8}\int_0^2 \rho^4 \int_0^{2\pi} d\theta \, d\rho \\ &= \frac{3\pi}{4} \int_0^2 \rho^4 \, d\rho = \frac{3\pi}{20}\left(2^5 - 0\right) = \boxed{\frac{24\pi}{5}.} \end{aligned}$$

9. Consider the function

$$f(x,y) = x^2 + xy + y^3 + 2.$$

(a) At the point $(2,-1)$, in which direction should you move to produce the greatest rate of *decrease* in f?

Solution:
$\nabla f|_{(2,-1)} = \langle 2x+y, x+3y^2 \rangle|_{(2,-1)} = \langle 3, 5 \rangle$, so the direction of greatest decrease is

$$\boxed{-\nabla f|_{(2,-1)} = \langle -3, -5 \rangle.}$$

(b) At the point $(2,-1)$, what is the directional derivative of f in the direction towards the origin?

Solution:
The direction we consider is $\mathbf{u} = \frac{1}{||\mathbf{v}||}\mathbf{v}$ with $\mathbf{v} = (0,0) - (2,-1) = (-2,1)$, so $\mathbf{u} = \langle -2/\sqrt{5}, 1/\sqrt{5} \rangle$. Then

$$D_{\mathbf{u}} f(2,-1) = \nabla f|_{(2,-1)} \cdot \mathbf{u} = \langle 3,5 \rangle \cdot \langle -2/\sqrt{5}, 1/\sqrt{5} \rangle = \boxed{-1/\sqrt{5}.}$$

(c) Show that $(0,0)$ and $(-1/12,1/6)$ are critical points of f. (They are the only critical points, but you need not show that.)

Solution:

Since $\nabla f = \langle 2x+y, x+3y^2\rangle$, we see $\boxed{\nabla f|_{(0,0)} = \langle 0,0\rangle}$ and $\boxed{\nabla f|_{(-1/12,1/6)} = \langle 0,0\rangle}$.

(d) Determine whether each of the critical points is a maximum, a minimum, or a saddle point.

Solution:

Applying the Second Partials Test, we consider the matrix of second partial derivatives,

$$\begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 6y \end{pmatrix}.$$

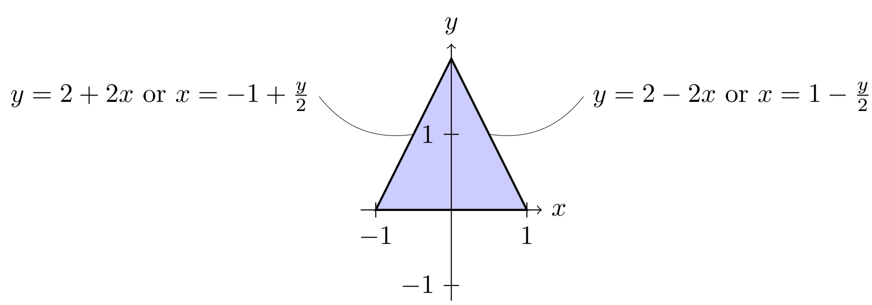
At $(0,0)$, the determinant is $-1<0$, so that point is a $\boxed{\text{saddle}}$.

At $(-1/12,1/6)$, the determinant is $1>0$, and the upper left entry is $2>0$, so that point is a $\boxed{\text{minimum}}$.

10. Let $(\overline{x},\overline{y})$ be the center of mass of a triangular planar lamina of density $\rho(x,y)=y$ determined by the vertices $(-1,0)$, $(1,0)$, and $(0,2)$.

(a) Write an integral formula for $\overline{x}$. Fully set up the integral(s) with the integrand and limits of integration, but DO NOT EVALUATE.

Solution: The region of integration is:



So we have:

$$\overline{x} = \frac{\iint_R x\rho(x,y)\,dA}{A} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

$$= \boxed{\frac{\int_0^2 \int_{-1+\frac{y}{2}}^{1-\frac{y}{2}} xy\,dx\,dy}{\int_0^2 \int_{-1+\frac{y}{2}}^{1-\frac{y}{2}} y\,dx\,dy} \quad \text{or} \quad \frac{\int_{-1}^0 \int_0^{2+2x} xy\,dy\,dx + \int_0^1 \int_0^{2-2x} xy\,dy\,dx}{\int_{-1}^0 \int_0^{2+2x} y\,dy\,dx + \int_0^1 \int_0^{2-2x} y\,dy\,dx}}.$$

(b) Can you find the value of $\overline{x}$ without computation? Explain your answer.

Solution: Yes. We have that $\boxed{\overline{x} = 0 \text{ by symmetry}}$ of the region and because $\rho(x,y)$ is independent of x.