MATH 253 Name: ______________________

Midterm 1 February 4, 2022

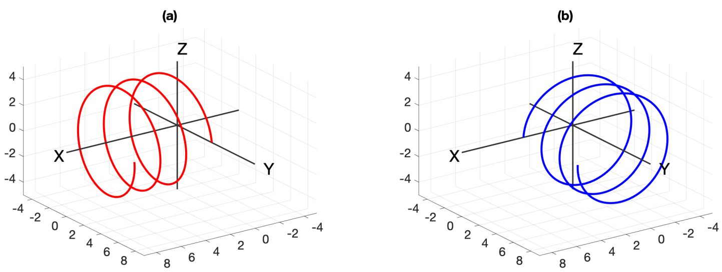
Instructions: 100 points total. Use only your brain and writing implement. You have 70 minutes to complete this exam. Answers should be given in 'good' mathematical form (simplified, etc.) There are five pages of problems, so 14 minutes a page. If you can not do a problem, move on. Good luck.

1. (26 pts.) The trajectory of a particle in $\mathbb{R}^3$ is given by the equation

$$\mathbf{r}(t) = \langle 4\cos(3t),\, t,\, 4\sin(3t)\rangle$$

where t is measured in seconds and $\mathbf{r}(t)$ in meters.

(a) (5 pts.) Two possible plots of this trajectory are drawn. Is the correct plot given by that in (a) or (b)? Justify your choice.



Answer:

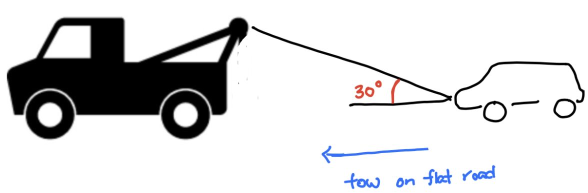
(b) (7 pts.) Give the tangent vector to the curve at the time $t = \frac{3\pi}{4}$. Then on the (correct) plot above, sketch and label the location of the particle (as a point) and the tangent vector at the time $t = \frac{3\pi}{4}$. (Do not worry about getting the magnitude correct.)

(c) (6 pts.) Give the speed of the particle at the time $t = \frac{3\pi}{4}$, including units in your final answer.

(continued on next page.)

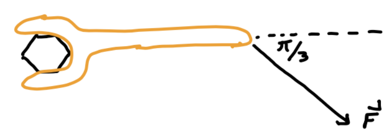
(d) (8 pts.) Find the distance traveled by the particle between the time $0 \leq t \leq \frac{\pi}{3}$, including units in your final answer.

2. (10 pts.) Your car breaks down and is pulled for a distance of 5 kilometers by a tow truck on a flat road. See Figure. Determine the amount of work done in towing your car if a constant force of magnitude 300 Newtons on the tow chain is applied. Include units in your final answer.



3. (10 pts.) A bolt is tightened by applying a 20-Newton force to a .25 meter wrench as shown. (The labeled angle is $\frac{\pi}{3}$.)

(a) (7 pts.) Find the magnitude of the torque vector, $|\boldsymbol{\tau}|$, about the center of the bolt. Include units in your answer.



(b) (3 pts.) In what direction does the torque vector $\boldsymbol{\tau}$ point?

4. (22 pts.) Consider the two vectors in $\mathbb{R}^3$:

$$\mathbf{a} = \langle -1,\ 2,\ 5\rangle \qquad \text{and} \qquad \mathbf{b} = \langle 3,\ -2,\ 1\rangle$$

(a) (4 pts.) Determine if the angle between the two vectors is acute, right, or obtuse. Briefly justify your answer.

(b) (5 pts) Give the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ and find its length.

Answer: $\mathbf{proj_a}(\mathbf{b}) =$ ______________________________ and its length is ________________

(c) (8 pts) Noting that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$, find the equation of the plane that is parallel to the plane containing the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ *AND* passes through the point $P(2, 0, -1)$.

(d) (5 pts) Find the area of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$.

5. (20 pts) Consider the two planes given by equations:

$$\begin{array}{lrcrcrcl} \text{Plane 1:} & x & + & 2y & - & z & = & 3 \\ \text{Plane 2:} & 3x & - & y & + & z & = & 2 \end{array}$$

(a) (5 pts.) Carefully prove that the two planes are not parallel.

(b) (5 pts.) Find the angle θ between the two planes.

(c) (10 pts.) Find the equation of the line $\ell(t)$ of intersection of the two planes. (You may give your answer in vector or parametric form.)

Answer: $\ell(t) =$ ________________________________ .

6. (12 pts.) An atom moves on an electrified plate with acceleration given by

$$\mathbf{a}(t) = \left\langle e^{\cos(t)}\sin(t) + 2,\ \left(t - \frac{\pi}{2}\right)^2 \right\rangle \quad \text{ft/s}^2$$

and velocity at time $t = \frac{\pi}{2}$ is

$$\mathbf{v}\left(\frac{\pi}{2}\right) = \langle \pi + 3,\ -1 \rangle .$$

Give the velocity of this atom at all times t. Include units in your final answer.

FORMULAS

The formulas for curvature $\kappa(t)$ for a space curve are:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \qquad\qquad \kappa = \frac{d\mathbf{T}}{d\mathbf{s}}$$