MATH202X-F01-UX1 Midterm Exam 1 Name: Answer Key

Spring 2015

Instructions. You have 60 minutes. No book, notes or calculators allowed. *Show all your work* in order to receive full credit.

1. A parallelepiped has three edges given by the vectors

$$\mathbf{a} = \langle -2, 1, 1\rangle, \ \mathbf{b} = \langle 1, 2, 1\rangle, \quad \mathbf{c} = \langle 2, -1, 3\rangle.$$

Find:

(a) The area of the face with edges $\mathbf{a}$ and $\mathbf{b}$.

Solution:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -1\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$$

$$\text{Area} = ||\mathbf{a} \times \mathbf{b}|| = \sqrt{(-1)^2 + 3^2 + (-5)^2} = \boxed{\sqrt{35}}$$

(b) The volume of the parallelepiped.

Solution:

$$\text{Volume } = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = |2(-1) + (-1)3 + 3(-5)| = |-20| = \boxed{20}$$

2. Let $\mathbf{r_1}(t) = \langle \ln t, 1 - \sqrt{5-t}\rangle$ and $\mathbf{r_2}(t) = \langle \sin(t-2), 1-t\rangle$ represent the trajectories of two particles in the plane.

(a) What is the domain of $\mathbf{r_1}$?

Solution:

$$\boxed{0 < t \leq 5, \text{ or } (0, 5]}$$

(b) Show that $\langle 0, -1\rangle$ is a point of intersection of the two trajectories. Do the particles collide there? Briefly explain your answer.

Solution: $\boxed{\mathbf{r_1}(t) = \langle 0, -1\rangle \text{ precisely when } t = 1, \text{ while } \mathbf{r_2}(t) = \langle 0, -1\rangle \text{ precisely when } t = 2.}$ So we have that $\langle 0, -1\rangle$ is on both trajectories, but the particles pass through that point at different times. Therefore, $\boxed{\text{they do not collide there.}}$

(c) Find the velocity of the first particle at $t = 1$.

Solution:

$$\mathbf{r}'(t) = \left\langle \frac{1}{t}, -\frac{1}{2}(5-t)^{-1/2}(-1) \right\rangle = \left\langle \frac{1}{t}, \frac{1}{2\sqrt{5-t}} \right\rangle$$

$$\boxed{\mathbf{r}'(1) = \left\langle 1, \frac{1}{4} \right\rangle}$$

3. A plane E is given by $x - y - z = 1$, and three points by $P(2, -2, -1)$, $Q(3, -2, 0)$, and $R(1, 2, -1)$.

(a) How far is the point P from the plane E?

Solution: Pick any point S on the plane E, say $S(1, 0, 0)$. Then $\overrightarrow{PS} = \langle -1, 2, 1\rangle$. Projecting this vector onto the normal to the plane E, $\mathbf{n} = \langle 1, -1, -1\rangle$ yields

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{PS} = \frac{\overrightarrow{PS} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} = \frac{-4}{3} \mathbf{n}.$$

The distance is

$$\|\operatorname{proj}_{\mathbf{n}} \overrightarrow{PS}\| = \frac{4}{3} \|\mathbf{n}\| = \frac{4\sqrt{3}}{3}.$$

Or using directly the distance formula if we rewrite E as: $x - y - z - 1 = 0$:

$$d(P, E) = \frac{|(2) - (-2) - (-1) - 1|}{\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{4}{\sqrt{3}} = \boxed{\frac{4\sqrt{3}}{3}.}$$

(b) Give a parameterization of the line through R that is parallel to the one through P and Q.

Solution: Since $\overrightarrow{PQ} = \langle 1, 0, 1\rangle$, a parametrization is

$$\mathbf{r}(t) = \langle 1, 2, -1\rangle + t\,\langle 1, 0, 1\rangle = \boxed{\langle 1 + t, 2, -1 + t\rangle\,.}$$

(c) Is the plane through the three points P, Q, and R orthogonal to the plane E?

Solution: Since $\overrightarrow{PQ} = \langle 1, 0, 1\rangle$, $\overrightarrow{PR} = \langle -1, 4, 0\rangle$, the normal to the plane of P, Q, R is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ -1 & 4 & 0 \end{vmatrix} = -4\mathbf{i} - 1\mathbf{j} + 4\mathbf{k} = \langle -4, -1, 4\rangle\,.$$

The plane E has normal $\langle 1, -1, -1\rangle$.
Since $\langle 1, -1, -1\rangle \cdot \langle -4, -1, 4\rangle = -7 \neq 0$, $\boxed{\text{the planes are not orthogonal.}}$

4. A particle is moving in space with the following ***velocity*** (measured in ft/s):

$$\mathbf{v}(t) = \left\langle t^2, 2t, 0\right\rangle.$$

(a) Find the position of the particle $\mathbf{r}(t)$ if $\mathbf{r}(3) = \langle 2, 1, 5\rangle$.

Solution: Since $\mathbf{r}'(t) = \left\langle t^2, 2t, 0\right\rangle$, antidifferentiation yields $\mathbf{r}(t) = \left\langle \frac{t^3}{3} + c, t^2 + d, e\right\rangle$ for constants c, d, e. but

$$\langle 2, 1, 5\rangle = \mathbf{r}(3) = \langle 9 + c, 9 + d, e\rangle\,,$$

so $c = -7$, $d = -8$, and $e = 5$. Thus

$$\boxed{\mathbf{r}(t) = \left\langle \frac{t^3}{3} - 7, t^2 - 8, 5\right\rangle.}$$

(b) Find the distance traveled by the particle (i.e. the arc length) between $t = 0$ s and $t = 2$ s.

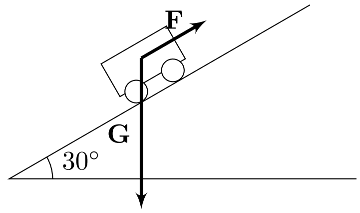
Solution: Since

$$\text{Arc length} = \int_0^2 \|\mathbf{v}(t)\| dt = \int_0^2 \sqrt{t^4 + 4t^2} dt = \int_0^2 t\sqrt{t^2 + 4} dt,$$

substituting $u = t^2 + 4$, $du = 2t\,dt$,

$$\text{Arc length} = \frac{1}{2} \int_4^8 u^{1/2}\,du = \frac{1}{2}\frac{2}{3} u^{3/2}\Big|_4^8 = \frac{1}{3}(8^{3/2} - 4^{3/2}) = \boxed{\frac{8}{3}\left(2\sqrt{2} - 1\right) \text{ ft.}}$$

5. A small cart (with frictionless wheels) weighing 20 lbs is on a 30 foot ramp which makes a 30° angle with the horizontal, as shown in the figure. A force **F** is applied to the cart by a rope going uphill, parallel to the ramp. Let **G** denote the force of gravity on the cart.



(a) Find the vector component of **G** along the ramp, and the vector component perpendicular to the ramp. (Be sure you indicate which is which.)

Solution: $\mathbf{G} = \langle 0 - 20 \rangle$, and the direction of the ramp is $\mathbf{u} = \left\langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$. The component of **G** in this direction is

$$\operatorname{proj}_{\mathbf{u}} \mathbf{G} = \frac{\mathbf{G}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{-10}{1}\mathbf{u} = \boxed{\left\langle -5\sqrt{3}, -5 \right\rangle.}$$

The component of **G** perpendicular to the ramp is

$$\mathbf{G} - \operatorname{proj}_{\mathbf{u}} \mathbf{G} = \langle 0, -20 \rangle - \left\langle -5\sqrt{3}, -5 \right\rangle = \boxed{\left\langle 5\sqrt{3}, -15 \right\rangle.}$$

(b) If **F** has magnitude 8 lbs, in which direction will the cart move? Explain your answer.

Solution: Since $||\operatorname{proj}_{\mathbf{u}} \mathbf{G}|| = ||\left\langle -5\sqrt{3}, -5 \right\rangle|| = \sqrt{75+25} = 10$, the force which would cause the cart to move downhill has magnitude 10 lbs. An opposing force of 8 lbs will not overcome this, so $\boxed{\text{the cart will move downhill.}}$

Alternate approach: $\mathbf{F} = 8\left\langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \right\rangle = \left\langle 4\sqrt{3}, 4 \right\rangle$, so $\operatorname{proj}_{\mathbf{u}} \mathbf{G} + \mathbf{F} = \left\langle -\sqrt{3}, -1 \right\rangle$. Thus the net force is downhill, and that is the direction the cart will move.

(c) When the cart is at the top of the ramp, the rope suddenly snaps and the cart rolls to the bottom. How much work is done by gravity? Give units.

Solution: $W = \mathbf{G}\cdot\mathbf{d}$, where $\mathbf{d} = -30\left\langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \right\rangle$ is the displacement vector. So

$$W = \langle 0, -20 \rangle \cdot \left\langle -15\sqrt{3}, -15 \right\rangle = \boxed{300 \text{ ft lbs.}}$$

6. Consider $\mathbf{r}(t) = \left\langle t^2 + 1, 2t \right\rangle$.

(a) Show that the underlying curve satisfies the equation: $4(x-1) = y^2$.

Solution: Inserting $x = t^2 + 1,\ y = 2t$ into the equation yields

$$4\left((t^2+1) - 1\right) = (2t)^2,$$

which is obviously true.

(b) Find the unit tangent vector to the parameterized curve $\mathbf{r}(t)$ at all times t.

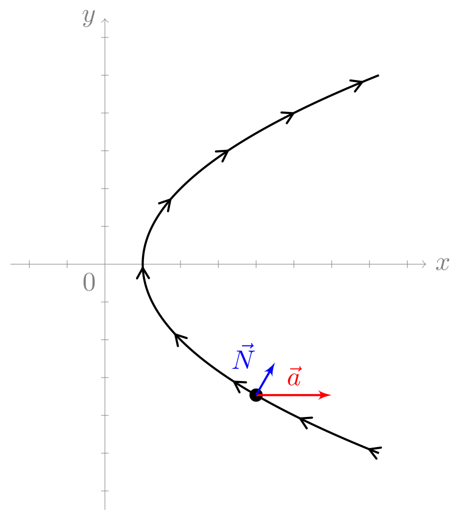
Solution:

$$\mathbf{r}'(t) = \langle 2t, 2 \rangle = 2\langle t, 1 \rangle, \quad ||\mathbf{r}'(t)|| = 2\sqrt{t^2+1}$$

$$\mathbf{T} = \frac{1}{||\mathbf{r}'(t)||}\mathbf{r}'(t) = \frac{\cancel{2}\langle t, 1 \rangle}{\cancel{2}\sqrt{t^2+1}} = \boxed{\left\langle \frac{t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle}$$

(c) The graph of $4(x-1) = y^2$ is shown below as well as the terminal point of $\mathbf{r}(-\sqrt{3})$.

Solution:



(i) Orient the curve to represent the direction of motion of $\mathbf{r}(t)$;

(ii) Compute the acceleration when $t = -\sqrt{3}$. Sketch it above, at the point shown, labelling it $\vec{a}$;

Solution:

$$\mathbf{r}''(t) = \langle 2, 0 \rangle$$

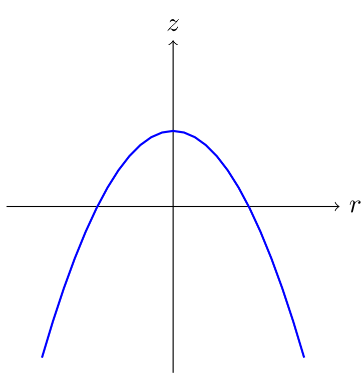
$$\boxed{\mathbf{r}''(-\sqrt{3}) = \langle 2, 0 \rangle}$$

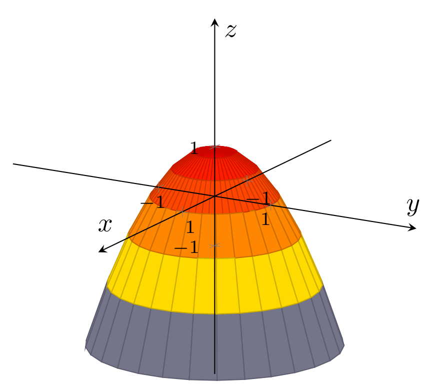
(iii) Sketch the principal unit normal vector at $t = -\sqrt{3}$ above, at the point shown, labelling it $\vec{N}$. (No computation is necessary if you understand its geometric meaning.)

7. Sketch 3-dimensional graphs of the following equations, given in either rectangular, cylindrical, or spherical coordinates. Use the given axes, choose and indicate your scale.

(a) $z = 1 - r^2$

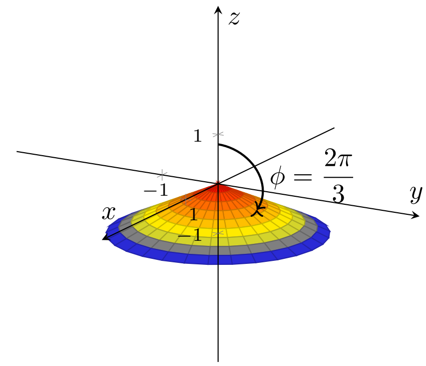
Solution:





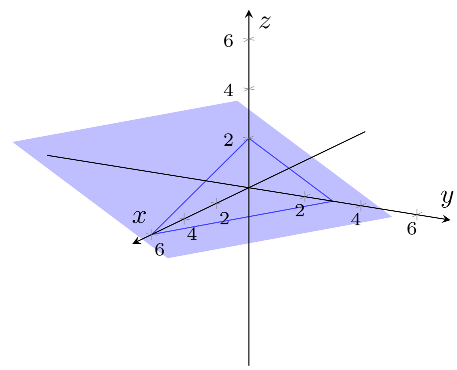
(b) $\phi = 2\pi/3$

Solution:



(c) $x + 2y + 3z = 6$

Solution:



(d) $z = \frac{1}{y}$

Solution: