Math F253 Midterm 1 Fall 2022

Name: Solutions

Section: F02 (Maxwell)

Student Id: ____________

Calculator Model: ____________

Rules:

You have 70 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	12	
3	12	
4	16	
5	5	
6	10	
Extra Credit	3	
Total	65	

1. (10 points)

An object has acceleration

$$\mathbf{a}(t) = \langle 7, -2t, \pi \sin(\pi t) \rangle$$

and has velocity $\mathbf{v}(0) = \langle 1, 2, 1 \rangle$.

a. Determine $\mathbf{v}(t)$.

$$\vec{v}(t) = \int \vec{a}(t)\, dt + \vec{c}$$

$$= \langle 7t, -t^2, -\cos(\pi t) \rangle + \vec{c}$$

$$\left.\begin{array}{l} \vec{v}(0) = \langle 0, 0, -1 \rangle + \vec{c} \\ \text{and } \vec{v}(0) = \langle 1, 2, 1 \rangle \end{array}\right] \Rightarrow \vec{c} = \langle 1, 2, 2 \rangle$$

$$\boxed{\vec{v}(t) = \langle 1+7t, 2-t^2, 2-\cos(\pi t) \rangle}$$

b. Determine all moments t such that the object's velocity is perpendicular to the vector $\mathbf{w} = \langle 1, 1, 0 \rangle$.

$$\vec{v}(t) \cdot \vec{w} = 0$$

$$\Rightarrow (1+7t)\cdot 1 + (2-t^2)\cdot 1 + (2-\cos(\pi t))\cdot 0 = 0$$

$$\Rightarrow 1+7t+2-t^2 = 0$$

$$\Rightarrow t^2 - 7t - 3 = 0$$

$$\boxed{t = \frac{7 \pm \sqrt{49+12}}{2} = \frac{7 \pm \sqrt{61}}{2} = 7.41, -0.41}$$

2. (16 points)

Consider the vector-valued function

$$\mathbf{r}(t) = \langle t, \sin(2t), \cos(2t) \rangle .$$

a. Compute $\mathbf{r}(\pi/2)$ and $\mathbf{r}'(\pi/2)$.

$$\boxed{\vec{r}(\pi/2) = \left\langle \frac{\pi}{2}, \sin(\pi), \cos(\pi) \right\rangle = \left\langle \frac{\pi}{2}, 0, -1 \right\rangle}$$

$$\vec{r}\,'(t) = \langle 1, 2\cos(2t), -2\sin(2t) \rangle$$

$$\boxed{\vec{r}\,'(\pi/2) = \langle 1, 2\cos(\pi), -2\sin(\pi) \rangle = \langle 1, -2, 0 \rangle}$$

b. Determine a vector that points in the direction of $\mathbf{r}'(\pi/2)$ but that has length 7.

$$\|\vec{r}\,'(\pi/2)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\text{vector: } \frac{7}{\sqrt{5}}\,\vec{r}\,'(\pi/2) = \boxed{\left\langle \frac{7}{\sqrt{5}}, \frac{-14}{\sqrt{5}}, 0 \right\rangle}$$

c. Determine the equation of a line that passes through the point $\mathbf{r}(\pi/2)$ and that is parallel to the vector $\mathbf{r}'(\pi/2)$. Either the symmetric form of the line or a parameterized line is acceptable.

$$\ell(s) = \left\langle \frac{\pi}{2}, 0, -1 \right\rangle + s\langle 1, -2, 0 \rangle$$

$$= \boxed{\left\langle \frac{\pi}{2} + s, -2s, -1 \right\rangle}$$

d. Determine the length of the original curve $\mathbf{r}(t)$ between the parameters $t = 0$ and $t = \pi/2$.

$$\|\vec{r}\,'(t)\| = \left(1^2 + 4\cos^2(2t) + 4\sin^2(2t)\right)^{1/2}$$

$$= \sqrt{5}$$

$$\text{length: } \int_0^{\pi/2} \|\vec{r}\,'(t)\|\,dt = \int_0^{\pi/2} \sqrt{5}\,dt = \boxed{\frac{\sqrt{5}}{2}\pi}$$

3. (12 points)

Consider two objects following the following the following trajectories as a function of time t:

$$\mathbf{r}(t) = \langle e^t, \sin(5t), e^{-t} \rangle$$
$$\mathbf{s}(t) = \left\langle te^{2t}, e^{-t}, 4t \right\rangle.$$

a. How far apart are the objects at time $t = 0$?

$\vec{r}(0) = \langle 1, 0, 1 \rangle$

$\vec{s}(0) = \langle 0, 1, 0 \rangle$

$\|\vec{r}(0) - \vec{s}(0)\| = \|\langle 1, -1, 1 \rangle\| = \boxed{\sqrt{3}}$

b. What is the angle between the velocities of the two objects at time $t = 0$?

$\vec{r}\,'(t) = \langle e^t, 5\cos(5t), -e^{-t} \rangle$

$\vec{r}\,'(0) = \langle 1, 5, -1 \rangle$, $\|\vec{r}\,'(0)\| = \sqrt{27}$

$\vec{s}\,'(t) = \langle e^{2t} + 2te^{2t}, -e^{-t}, 4 \rangle$

$\vec{s}\,'(0) = \langle 1, -1, 4 \rangle$, $\|\vec{s}\,'(0)\| = \sqrt{18}$

$\vec{r}\,'(0) \cdot \vec{s}\,'(0) = 1 - 5 - 4 = -8$

$\theta = \arccos\left(\dfrac{\vec{r}\,'(0) \cdot \vec{s}\,'(0)}{\|\vec{r}\,'(0)\|\ \|\vec{s}\,'(0)\|}\right) = \boxed{\arccos\left(\dfrac{-8}{\sqrt{27}\sqrt{18}}\right) = 111.3^\circ}$

c. What are the the speeds of the two objects at time $t = 0$? **Hint:** I didn't give you much space!

$\|\vec{r}\,'(0)\| = \sqrt{27}$ $\|\vec{s}\,'(0)\| = \sqrt{18}$

4. (12 points)

Consider the following three points

$$P(1,3,2), \quad Q(4,-1,1), \quad R(2,6,2)$$

a. Determine the equation of the plane that passes through the three points

$$\vec{PQ} = \langle 3, -4, -1 \rangle, \quad \vec{PR} = \langle 1, 3, 0 \rangle$$

$$\vec{n} = \vec{PQ} \times \vec{PR}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & -1 \\ 1 & 3 & 0 \end{vmatrix} = 3\hat{i} - (+1)\hat{j} + 13\hat{k}$$

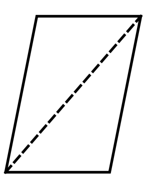
$$= \langle 3, -1, 13 \rangle$$

$$\boxed{\begin{gathered} 3(x-1) - (y-3) + 13(z-2) = 0 \\ \text{or} \\ 3x - y + 13z = 26 \end{gathered}}$$

b. Find the equation of the parallel plane that passes through the origin.

$$3x - y + 13z = 0$$

c. Find the area of the triangle spanned by the three points. **Hint:**

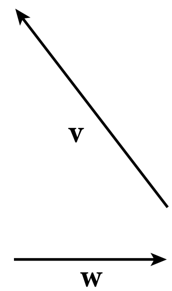


$$\frac{\|\vec{PQ} \times \vec{PR}\|}{2} = \frac{\|\vec{n}\|}{2} = \frac{\sqrt{9+1+13^2}}{2} \boxed{= \frac{\sqrt{179}}{2}}$$

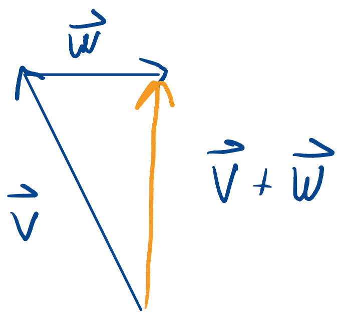
$$= 6.69\ldots$$

5. (5 points)

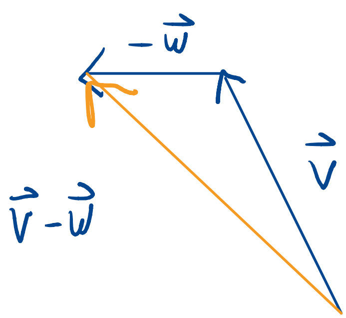
Consider the vectors **v** and **w** sketched below.



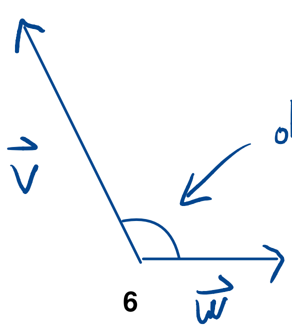
a. Sketch the vector **v** + **w**. Your sketch should contain enough details to know how your answer was arrived at.



b. Sketch the vector **v** - **w**. Your sketch should contain enough details to know how your answer was arrived at.



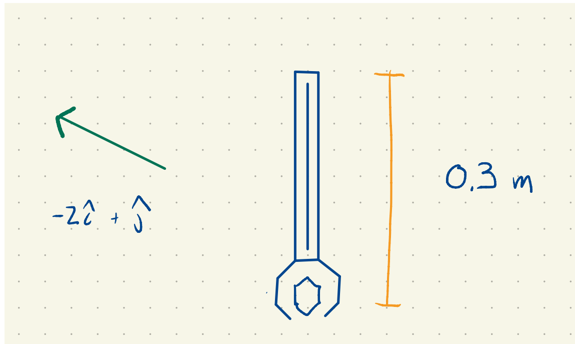
c. Is the dot product between **v** and **w** positive, negative or zero? Justify your answer.



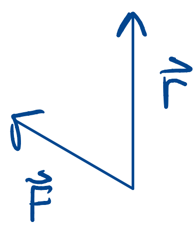
obtuse $\Rightarrow$ $\boxed{\vec{v} \cdot \vec{w} < 0}$

6. (10 points)

A wrench has its mouth on a bolt at the origin and its other end at the position $\langle 0, 0.3\rangle$ meters.



a. A force vector $\mathbf{F}$ is being applied in the direction $-2\mathbf{i} + \mathbf{j}$. Is the resulting torque pointing into or out of the page?



right had rule

$\Rightarrow$ $\vec{r} \times \vec{F}$ points out of page

b. Determine what the force vector $\mathbf{F}$ (still pointing in the direction $-2\mathbf{i} + \mathbf{j}$) should be to obtain a total torque of 60 Nm.

$\vec{F} = \langle -2A, A, 0\rangle$ for some A

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0.3 & 0 \\ -2A & A & 0 \end{vmatrix} = \hat{k}\,(2 \cdot A \cdot (0.3))$$

$$= 0.6A$$

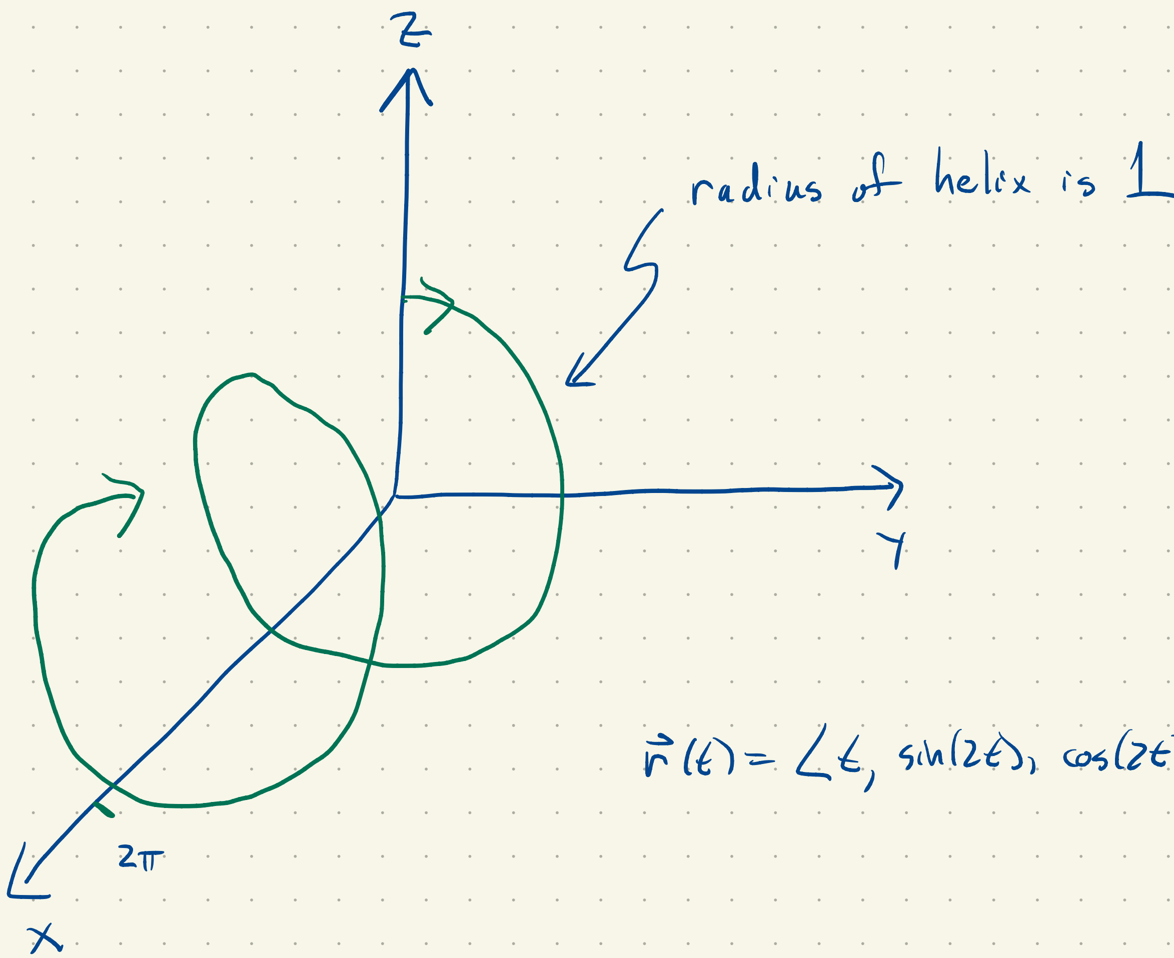
$\|\vec{\tau}\| = 0.6A$

Want $\|\vec{\tau}\| = 60 \Rightarrow$ $0.6A = 60 \Rightarrow A = 100$

$$\boxed{\vec{F} = \langle -200, 100, 0\rangle \text{ N}}$$

7. (Extra Credit: 3 points)

On the back side of this page, sketch the curve from problem ~~8~~ 2. For full credit your sketch must indicate the directions of the x, y and z axes.



$\vec{r}(t) = \langle t, \sin(2t), \cos(2t) \rangle$