MATH 253
Final Exam

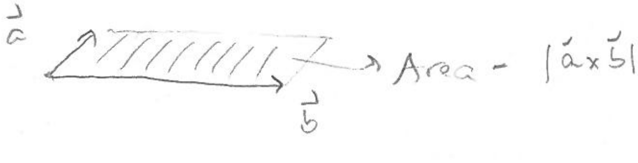
Name: SOLUTIONS

April 27, 2022

Instructions: 110 points total plus an extra credit problem. You get your score out of 100 points. Use only your brain, a writing implement, and a single formula sheet. Your formula sheet, with your name on it, must be turned in with your exam. You have 120 minutes to complete this exam which is 9 pages in length (7 pages of math questions, one blank page, one Integral Guide page). Answers should be given in 'good' mathematical form (simplified, etc.) Bald answers will receive little, if any, credit. If you can not do a problem, move on. Good luck and have a nice summer.

1. (5 pts.) Given two non-parallel vectors **a** and **b** in $\mathbb{R}^3$, what is the geometric meaning of $|\mathbf{a}\times\mathbf{b}|$?

$|\vec{a}\times\vec{b}|$ = area of parallelogram spanned by $\vec{a}$ and $\vec{b}$



2. (10 pts.) Consider the vector field

$$\mathbf{F}(x,y,z) = \mathbf{i} + (x+yz)\mathbf{j} + (xy-\sqrt{z})\mathbf{k}$$

(a) (4 pts.) Compute curl **F**.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy-z^{\frac{1}{2}} \end{vmatrix} = (x-y)\hat{i} - (y-0)\hat{j} + (1-0)\hat{k}$$

$$= \boxed{\langle x-y, -y, 1\rangle}$$

(b) (3 pts.) Compute div **F**.

$$\nabla\cdot\vec{F} = 0 + z - \tfrac{1}{2}z^{-\frac{1}{2}} = \boxed{z - \frac{1}{2\sqrt{z}}}$$

(c) (3 pts.) Suppose the vector field **F** represents the velocity field for some fluid. Compute the divergence of **F** at the point $(1,1,1)$, and indicate what div $\mathbf{F}(1,1,1)$ tells you about the net fluid flow at $(1,1,1)$.

$$\text{div}\,\vec{F}(1,1,1) = 1 - \frac{1}{2\sqrt{1}} = \boxed{\frac{1}{2}}$$

Since $\text{div}\,\vec{F}(1,1,1) > 0$

$\boxed{\text{net outward flow}}$

at this point

3. (10 pts.) Suppose you are climbing up out of a canyon whose shape is given by

$$z = -100 + .05x^2 + .01x + .02y^2 \quad \text{meters},$$

and your current position in this canyon is given by the point with coordinates $(10, 20, -86.9)$. Assume that the positive x-axis points due east and the positive y-axis points due north.

(a) (7 pts.) If you walk due south $\to \langle 0,-1\rangle$ from your current coordinates of $(x, y) = (10, 20)$, will you start to ascend or descend? At what rate? (A complete answer gives a brief justification.)

$D_{\vec u}(10,20) = \nabla z(10,20) \cdot \langle 0,-1\rangle$

$\nabla z = \langle .1x + .01, .04y\rangle$

$\nabla z(10,20) = \langle 1.01, .8\rangle$

$= \langle 1.01, 0.8\rangle \cdot \langle 0,-1\rangle$

$= \boxed{-.8}$ → rate. negative means $\boxed{\text{descending}}$

(b) (3 pts.) In what direction should you walk from your current coordinates of $(x, y) = (10, 20)$ to climb in the direction of steepest ascent? Briefly justify your answer.

$\nabla z(10,20) = \langle 1.01, .8\rangle$

∇z gives direction from $(10,20)$ of maximal increase (= ascent)

4. (10 pts.) Use Green's Theorem to compute

$$\oint_C -yx^2\, dx + xy^2\, dy,$$

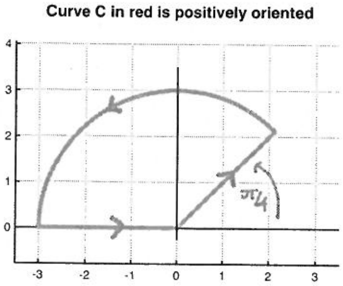
where C is the simple closed curve shown.

$P = -yx^2 \quad Q = xy^2$

$\frac{\partial Q}{\partial x} = y^2 \quad \frac{\partial P}{\partial y} = -x^2$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x^2 + y^2$

Curve C in red is positively oriented



$\oint_C P\,dx + Q\,dy$

$= \iint x^2 + y^2\, dA = \int_{\pi/4}^{\pi} \int_0^3 r^2 (r\,dr\,d\theta)$

$= \int_{\pi/4}^{\pi} \int_0^3 r^3\, dr\, d\theta = \frac{3\pi}{4}\left[\frac{1}{4} r^4\right]_0^3 = \boxed{\frac{243\pi}{16}}$

5. (10 pts.) Give the linear approximation of the function $g(x, y) = 2 + \sin(4x + 3y)$ at the point $P(-3, 4)$.

$Z = g(-3,4) + \frac{\partial g}{\partial x}(-3,4)(x-(-3)) + \frac{\partial g}{\partial y}(-3,4)(y-4)$

$= 2 + 4(x+3) + 3(y-4)$

$= 2 + 4x + 3y$

$\boxed{Z = 2 + 4x + 3y}$

$g(-3,4) = 2$

$\frac{\partial g}{\partial x} = 4\cos(4x+3y)$

$\Rightarrow \frac{\partial g}{\partial x}(-3,4) = 4\cos(0)$

$= 4$

$\frac{\partial g}{\partial y} = 3\cos(4x+3y)$

$\Rightarrow \frac{\partial g}{\partial y}(-3,4) = 3\cos(0) = 3$

6. (10 pts. – 5 pts. each) Consider the function

$$f(x, y) = (\cos x)(\cos y)$$

on the square region R defined by $\boxed{\mathbf{0 < x < \pi,\quad 0 < y < \pi}}$.

(a) Find the single critical point of $f(x, y)$ within R. (Bald answers will receive no credit.)

$\frac{\partial f}{\partial x} = -\sin x \cos y$ $\quad$ $\frac{\partial f}{\partial y} = -\cos x \sin y$. Setting these equal to zero gives

$\boxed{-\sin x \cos y = 0}$ $\quad$ $\boxed{-\cos x \sin y = 0}$. But $\sin(\theta) \neq 0$ on $0 \leq \theta \leq \pi$, thus

$\downarrow$ $\quad$ $\downarrow$

$\cos y = 0$ $\quad$ $\cos x = 0$ $\quad$ and $x = \frac{\pi}{2}$ $\quad$ $y = \frac{\pi}{2}$

The critical point is $\boxed{\left(\frac{\pi}{2}, \frac{\pi}{2}\right)}$

(b) Use the Second Derivatives Test to determine if the critical point is a local maximum, local minimum, or saddle point, or there is not enough information to tell.

$f_{xy} = +\sin x \sin y \Rightarrow f_{xy}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = +1$

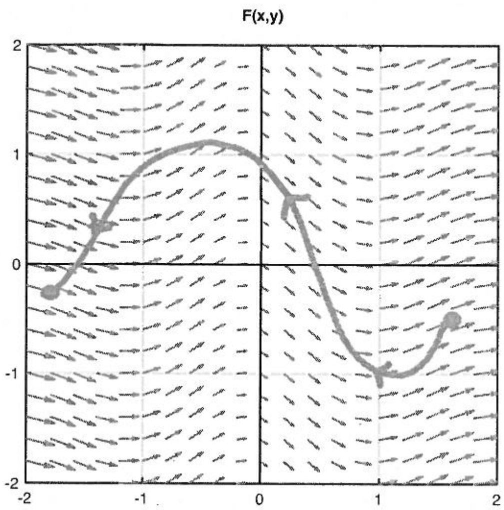
$f_{xx} = -\cos x \cos y$ $\quad$ $f_{xx}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 0$

$f_{yy} = -\cos x \cos y$ $\quad$ $f_{yy}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 0$

$) = \begin{vmatrix} -\cos x \cos y & \sin x \sin y \\ \sin x \sin y & -\cos x \cos y \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$\Rightarrow$ $\boxed{\text{SADDLE POINT}}$

7. (5 pts.) Consider the 2-dimensional force field $\mathbf{F}(\mathbf{x}, \mathbf{y})$ pictured below and the oriented curve C depicted in red. Is the work done by $\mathbf{F}$ in moving a particle along C negative, zero, or positive? Explain, including an interpretation of the appropriate integral in your brief mathematical explanation.



Negative:

$\vec{F}$ works "against" the motion of the particle along C.

8. (10 pts.) Among the points (x, y) satisfying the constraint $g: \quad y + 2x = 8$, use the Method of Lagrange multipliers to find all those points **maximizing** the function $f(x, y) = xy$ and the **maximum value** of $f(x, y)$ at these points.

$\nabla g = \langle 2, 1 \rangle$, $\nabla f = \langle y, x \rangle$, $\nabla f = \lambda \nabla g \Rightarrow \langle y, x \rangle = \langle 2\lambda, \lambda \rangle$

Three equations:

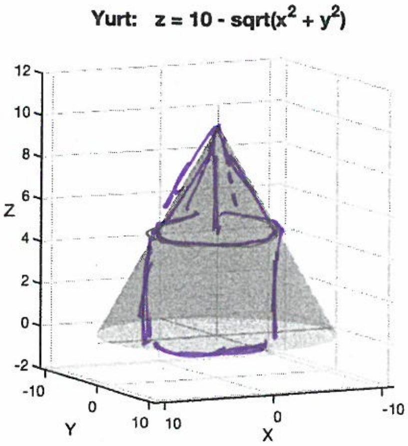
① $y = 2\lambda$ ② $x = \lambda$ ③ $y + 2x = 8$

Plugging ① and ② into ③: $(2\lambda) + 2(\lambda) = 8 \Rightarrow 4\lambda = 8 \Rightarrow \lambda = 2$

Thus, $y = 4$, $x = 2$ or c.p $= \boxed{(2,4)}$

The maximum value is $f(2,4) = 2 \cdot 4 = \boxed{8}$

9. (10 pts.) The inside of a yurt is shaped like the region between $z = 0$ and $z = 10 - \sqrt{x^2 + y^2}$, with $x^2 + y^2 \leq 25$, where x, y, z are measured in meters. If the density of mosquitoes inside the yurt is given by $\rho(x, y, z) = 12z$ mosquitoes$/m^3$, what is the total number of mosquitoes inside the yurt?



$\iiint_{Yurt} \rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^5\int_0^{10-r} 12z\, r\, dz\,dr\,d\theta$

$= \ldots$ 6375π mosquitoes

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If you followed the image to the left

$\int_0^{2\pi}\int_0^{10}\int_0^{10-r} 12z\, r\, dz\,dr\,d\theta = 10{,}000\pi$ mosquitoes

Too many!

10. (10 pts.) Let $\mathbf{F}(x, y, z) = \langle 2xy,\ z - y^2,\ x^2 + y^2 + z \rangle$ be a vector field in $\mathbb{R}^3$, and let S be a $2\times2\times2$ cube, centered at the origin, with edges parallel to the coordinate axes. Compute the flux of $\mathbf{F}$ through S. That is, compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$. (Hint: Use the Divergence Theorem.)

$\iint_S \vec{F} \cdot d\vec{S} = \iiint_{\square} \operatorname{div} \vec{F}\, dV = \iiint_{\square} 2y - 2y + 1\, dV = \iiint_{\square} dV$

$= \operatorname{Vol}(\square) = \boxed{8}$

11. (10 pts.) Consider the continuous vector field defined on all of $\mathbb{R}^2$,

$$\mathbf{F}(x,y) = \left\langle ye^{xy},\ xe^{xy} + \frac{1}{1+y} \right\rangle$$

and a curve C parameterized by

$$\mathbf{r}(t) = \langle t,\ 1-t^2 \rangle, \quad 0 \le t \le 1.$$

(a) (6 pts.) By finding ***all*** potential functions for $\mathbf{F}$, prove that $\mathbf{F}$ is conservative.

$f_x = ye^{xy}$ $\quad f_y = xe^{xy} + \frac{1}{1+y} \Rightarrow f(x,y) = e^{xy} + \ln|1+y| + C$

(b) (4 pts.) Use your answer in part (a) to compute the work done by $\mathbf{F}$ in moving a particle along the path C. That is, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

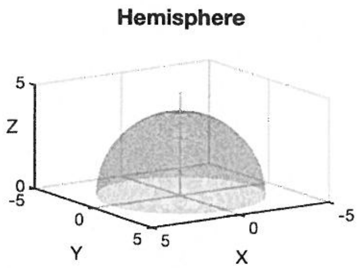
end point: $\vec{r}(1) = \langle 1, 0 \rangle$

beg point: $\vec{r}(0) = \langle 0, 1 \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = f(1,0) - f(0,1) = \left(e^0 + \ln|1|\right) - \left(e^0 + \ln|2|\right)$$

$$= \boxed{-\ln 2}$$

12. (10 pts. – 5 pts. each) Consider the plot of an upper hemisphere $z = \sqrt{16 - x^2 - y^2}$ shown below. The next questions involve both the surface S of the hemisphere, and the solid E it encloses above the xy-plane.



(a) Set up in **spherical coordinates** an iterated **triple integral** that computes the **volume of the solid hemisphere** under $z = \sqrt{16 - x^2 - y^2}$. A complete answer has limits of integration and is a triple integral. Do **not** compute the volume (particularly since there are easy non-Calculus ways to compute the volume).

$$\int_0^{2\pi}\int_0^{4}\int_0^{\pi/2} \rho^2 \sin\varphi \, d\varphi\, d\rho\, d\theta$$

(b) Now consider the **surface S** given by the hemisphere, $z = \sqrt{16 - x^2 - y^2}$. (This means the outside only, shown in darker blue in the figure.) By Stoke's theorem, the flux integral

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C=\partial S} \mathbf{F} \cdot d\mathbf{r},$$

where $C = \partial S$ is the boundary of the hemisphere S with a positive orientation. Use Stoke's theorem to compute the flux of $\mathbf{F} = \left\langle ye^z,\, y\sin(z),\, x\cos(z) \right\rangle$ through S, by using the right hand side of the the equation (the line integral).

(The next page is blank if you need additional space for your work.)

$0 \le t \le 2\pi$

$\vec{r}(t) = \langle 4\cos t, 4\sin t, 0 \rangle \qquad \vec{r}\,'(t) = \langle -4\sin t, 4\cos t, 0 \rangle$

$\sin(0)\ 4\sin t$

$$\iint_S \vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 4\sin t\, e^0, 0, 4\cos t\,(1) \rangle \cdot \langle -4\sin t, 4\cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} -16\sin^2 t \, dt = -16\int_0^{2\pi} \frac{1}{2} - \frac{\cos 2\theta}{2}\, d\theta = -16\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= -16(\pi) = \boxed{-16\pi}$$

(c) Extra credit: Compute the flux integral on the left hand side, showing all work. *Hint:* There are ways you could make this much *easier* on yourself.