Final Exam
Math 202 F01

Name: Solutions

12/14/12

1. Suppose that the temperature T, in degrees Celsius, in a certain region of space is given by the function

$$T(x, y, z) = 5x^2 - 3xy + yz,$$

where the position coordinates x, y, z are in meters.

(a) (5 pts.) What is the directional derivative at $(2, 0, 3)$ in the direction towards $(3, -2, 3)$? *Indicate units for your answer.*

Direction is $(3,-2,3)-(2,0,3) = \langle 1,-2,0\rangle$

So $\vec{u} = \frac{1}{\sqrt{1+4+0}}\langle 1,-2,0\rangle = \frac{1}{\sqrt{5}}\langle 1,-2,0\rangle$

$\nabla T\big|_{(2,0,3)} = \langle 10x-3y, -3x+z, y\rangle\big|_{(2,0,3)} = \langle 20,-3,0\rangle$

So $D_{\vec{u}}T = \nabla T\cdot\vec{u} = \langle 20,-3,0\rangle\cdot\frac{1}{\sqrt{5}}\langle 1,-2\rangle = \frac{26}{\sqrt{5}}\ \frac{{}^\circ C}{m}$

(b) (4 pts.) From the point $(2, 0, 3)$, in what direction should you begin moving to experience the greatest rate of *cooling*, and what would that rate be?

This is direction of greatest decrease of T, which is $-\nabla T = \langle -20,3,0\rangle$

The rate of decrease is the length of ∇T, i.e. $\sqrt{400+9+0} = \sqrt{409}\ \frac{{}^\circ C}{m}$

(or, if you prefer, $-\sqrt{409}$, since it is a decrease)

(c) (8 pts.) A straight wire is stretched from (2,0,3) to (1,1,1). Give an expression for the average temperature along the wire. Leave your answer in a form that a Calculus I student would understand; you do not need to completely evaluate any integrals.

Parameterization of wire: $\vec{r}(t) = \langle 2,0,3\rangle + t(\langle 1,1,1\rangle - \langle 2,0,3\rangle)$

$= \langle 2-t, t, 3-2t\rangle \qquad 0\le t\le 1$

$\vec{r}\,'(t) = \langle -1, 1, -2\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1+1+4} = \sqrt{6} \qquad ds = \sqrt{6}\,dt$

Average temp $= \dfrac{\int_C T\,ds}{\int_C ds} = \dfrac{\int_0^1 5(2-t)^2 - 3(2-t)t + t(3-2t)\ \sqrt{6}\,dt}{\int_0^1 \sqrt{6}\,dt}$

$= \dfrac{\cancel{\sqrt{6}}\int_0^1 (20-23t+6t^2)\,dt}{\cancel{\sqrt{6}}}$

2. (6 pts.) Find all critical points of $f(x, y) = x^3y + 12x^2 - 8y$, and, if possible, determine whether they are local maxima, local minima, or saddles.

$\nabla f = \vec{0}$: $\quad \langle 3x^2y + 24x, x^3 - 8 \rangle = \langle 0, 0 \rangle$

$$\left.\begin{array}{l} x^3 - 8 = 0 \Rightarrow x = 2 \\ 3x^2y + 24x = 0 \end{array}\right\} \Rightarrow 12y + 48 = 0 \Rightarrow y = -4$$

Only critical point is $(2, -4)$

2nd derivative test: $D = \begin{vmatrix} 6xy + 24 & 3x^2 \\ 3x^2 & 0 \end{vmatrix} = -9x^4$

$D(2,-4) < 0$ so $(2,-4)$ is a <u>saddle</u>

3. Consider the vector field $\mathbf{F} = \langle y^2 + 6ye^{3x}, 2e^{3x} + y + 2xy \rangle$.

(a) (4 pts.) This field is conservative. Find a potential function for $\mathbf{F}$.

$\frac{\partial f}{\partial x} = y^2 + 6ye^{3x}$

So $f(x,y) = xy^2 + 2ye^{3x} + c(y)$

$\frac{\partial f}{\partial y} = 2xy + 2e^{3x} + \frac{dc}{dy}(y) = 2e^{3x} + y + 2xy$

so $\frac{dc}{dy} = y$, $c(y) = \frac{y^2}{2} + D$

So $f(x,y) = xy^2 + 2ye^{3x} + \frac{y^2}{2} + D$

Note: D is optional, since <u>a</u> potential was asked for, not <u>all</u>

(b) (4 pts.) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the path parametrized by

$$\mathbf{r}(t) = \langle t\sin(\pi t), 2 + t\cos(\pi t) \rangle, \quad 0 \le t \le 4.$$

$\int_C \vec{F} \cdot d\vec{r} = f(\text{end}) - f(\text{start})$ with f from part (a)

end $= \vec{r}(4) = \langle 0, 6 \rangle$

start $= \vec{r}(0) = \langle 0, 2 \rangle$

So $\int_C \vec{F} \cdot d\vec{r} = f(0,6) - f(0,2) = (0 + 12 + 18) - (0 + 4 + 2) = 24$

(c) (2 pts.) If the field $\mathbf{F}$ represents a force, what is the physical interpretation of the integral you computed in part (b)? It is the <u>work</u> done by $\vec{F}$ on an object that moves along C.

4. A surface S is parameterized by $\mathbf{r}(u,v) = \langle u^2, u+v, u-v^2\rangle$, $0 \le u \le 2$, $0 \le v \le 4$.

(a) (4 pts.) Find an equation (of the form $ax+by+cz=d$) for the tangent plane to the surface at the point given by $u=1, v=2$.

Tangent vectors: $\vec{r_u} = \langle 2u, 1, 1\rangle \big|_{(1,2)} = \langle 2,1,1\rangle$

$\vec{r_v} = \langle 0, 1, -2v\rangle \big|_{(1,2)} = \langle 0,1,-4\rangle$

Normal vector: $\vec{r_u}\times\vec{r_v} = \langle 2,1,1\rangle \times \langle 0,1,-4\rangle = \langle -5, 8, 2\rangle$

point: $\vec{r}(1,2) = \langle 1, 3, -3\rangle$

plane: $-5(x-1)+8(y-3)+2(z+3)=0$ or $-5x+8y+2z=13$

(b) (6 pts.) Give an integral that would compute the flux of the vector field $\mathbf{F} = \langle 0, x, -y\rangle$ through S, oriented so that the normal vector has a positive z-component. You may leave your answer as an iterated integral, provided all that remains to be done is evaluation of it.

$$d\vec{S} = \vec{r_u}\times\vec{r_v}\,du\,dv = \langle 2u,1,1\rangle \times \langle 0,1,-2v\rangle\,du\,dv$$

$$= \langle -2v-1, 4uv, 2u\rangle\,du\,dv$$

so
$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^4\int_0^2 \langle 0, u^2, -(u+v)\rangle \cdot \langle -2v-1, 4uv, 2u\rangle\,du\,dv$$

$$= \int_0^4\int_0^2 u^2(4uv) - (u+v)2u\,du\,dv$$

$$= \int_0^4\int_0^2 (4u^3v - 2u^2 - 2uv)\,du\,dv$$

5. (7 pts.) Find all points satisfying the constraint $\underbrace{x^2+y^2}_{g(x,y)} = 1$ at which the function $f(x,y) = x^2 + y$ has its maximum value.

Lagrange multiplier: $\nabla f = \lambda \nabla g$

$\langle 2x, 1\rangle = \lambda\langle 2x, 2y\rangle$

So we must solve:

$$\begin{cases} 2x = \lambda 2x \\ 1 = \lambda 2y \\ x^2+y^2=1 \end{cases}$$

$2x = \lambda 2x \Rightarrow x=0$ or $\lambda = 1$

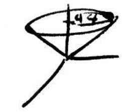
$x=0 \Downarrow y=\pm 1$; $\lambda=1 \Downarrow y=\frac{1}{2}, x=\pm\frac{\sqrt{3}}{2}$

so candidate points are $(0,\pm 1)$, $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Evaluating f at those 4 pts shows maxima are at $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

6. (8 pts.) Let S be the closed surface whose bottom is the cone $z = \sqrt{x^2+y^2}$ and whose top is the plane $z = 4$, oriented outward. Use Gauss's Divergence Theorem to compute the flux of the field

$$\mathbf{F} = \langle x + yz, x^2 + z^2, x^2 + z \rangle$$



through S.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_Q \nabla\cdot F\,dV = \iiint_Q (1+0+1)\,dV = 2\iiint_Q dV = 2(\text{vol of } Q)$$

Since Q is a cone with a circular base of radius 4 + height 4

$$\text{Vol } Q = \tfrac{1}{3}(\pi 4^2)(4) = \tfrac{64}{3}\pi, \quad \text{so} \quad \iiint_S \vec{F}\cdot d\vec{S} = \tfrac{128}{3}\pi$$

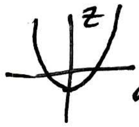
(If you prefer to compute vol Q, it is

$$\iiint_Q dV = \int_0^{2\pi}\int_0^4\int_r^4 r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^4 rz\Big|_{z=r}^4 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 4r - r^2\,dr\,d\theta = \int_0^{2\pi} 2r^2 - \tfrac{r^3}{3}\Big|_0^4 d\theta = \int_0^{2\pi}\left(32 - \tfrac{64}{3}\right)d\theta = (2\pi)\left(\tfrac{32}{3}\right) = \tfrac{64}{3}\pi \text{)}$$

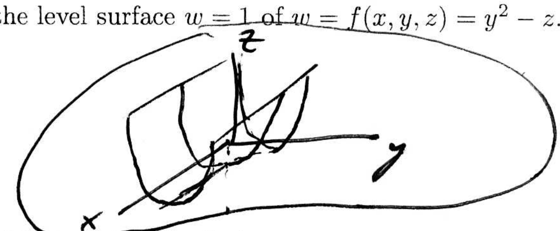
7. (6 pts. – 3 pts. each)

(a) Draw a rough sketch of the level surface $w = 1$ of $w = f(x, y, z) = y^2 - z$.



$1 = y^2 - z$

$z = y^2 - 1$



(b) At the point $(1, 2, 3)$ on this level surface, find a *unit* normal vector.

$$\underbrace{1 = y^2 - z}_{f(x,y,z)} \qquad \nabla f\Big|_{(1,2,3)} = \langle 0, 2y, -1\rangle\Big|_{(1,2,3)} = \langle 0, 4, -1\rangle$$

so $\vec{n} = \dfrac{1}{\sqrt{0+16+1}}\langle 0, 4, -1\rangle = \dfrac{1}{\sqrt{17}}\langle 0, 4, -1\rangle$

8. (6 pts.) An object's velocity vector at time t is given by $\mathbf{v}(t) = \langle t^2, \sin t, 2\rangle$, and its initial position at $t = 0$ is $\mathbf{r}(0) = \langle 1, 0, 2\rangle$. Give a formula for its position at all times.

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \tfrac{t^3}{3} + C, -\cos t + D, 2t + E \right\rangle$$

$$\langle 1, 0, 2\rangle = \vec{r}(0) = \langle C, -1 + D, E\rangle$$

so $C = 1$, $D = 1$, $E = 2$

$$\vec{r}(t) = \left\langle \tfrac{t^3}{3} + 1, 1 - \cos t, 2t + 2\right\rangle$$

9. (7 pts.) Evaluate the following integral, by first expressing it in a different coordinate system:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$$

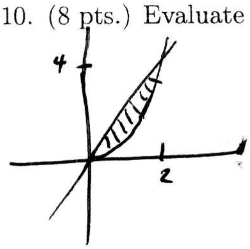
Convert to spherical coordinates:

$$\int_0^{\pi}\int_0^{\pi/2}\int_0^{3} (\rho\cos\phi)\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{\pi}\int_0^{\pi/2}\int_0^{3} \rho^4\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi}\int_0^{\pi/2} \frac{\rho^5}{5}\cos\phi\sin\phi\Big|_{\rho=0}^{3}\,d\phi\,d\theta = \frac{3^5}{5}\int_0^{\pi}\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\,d\theta$$

$$= \frac{3^5}{5}\int_0^{\pi} \frac{\sin^2\phi}{2}\Big|_0^{\pi/2}\,d\theta = \frac{3^5}{5}\left(\frac{1}{2}\right)\int_0^{\pi} d\theta = \frac{3^5\pi}{10} = \frac{243\pi}{10}$$

10. (8 pts.) Evaluate $\iint_R (x+1)\,dA$, where R is the region between the graphs of $y = x^2$ and $y = 2x$.



$$\iint_R (x+1)\,dA = \int_0^2\int_{x^2}^{2x}(x+1)\,dy\,dx = \int_0^2 xy+y\Big|_{y=x^2}^{2x}\,dx$$

$$= \int_0^2 (2x^2+2x-x^3-x^2)\,dx = \int_0^2 (x^2+2x-x^3)\,dx$$

$$= \frac{x^3}{3}+x^2-\frac{x^4}{4}\Big|_0^2 = \frac{8}{3}+4-4 = \frac{8}{3}$$

11. (15 pts. – 3 pts. each) Give short answers to the following:

(a) Give the equation of a plane through the point $(2, -1, 1)$ that is parallel to $3x - 2y + z = 1$.

$$3(x-2) - 2(y+1) + 1(z-1) = 0$$

or

$$3x - 2y + z = 9$$

(b) The area of a region R in the plane can be calculated by a line integral $\oint_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy$. Where does this formula come from? What is C here and in what direction should it be followed?

This is a consequence of Green's Thm, since $\frac{\partial}{\partial x}\left(\frac{x}{2}\right) + \frac{\partial}{\partial y}\left(\frac{-y}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1$

C is the curve bounding R, traced in a "counterclockwise" direction, i.e. move along C so if your head is in the z-direction, R is on your left.

(c) Is the angle between $\langle -1, 2, 3\rangle$ and $\langle 2, 3, -1\rangle$ acute ($< 90°$), right ($= 90°$), or obtuse ($> 90°$)? Show your work.

$\langle -1,2,3\rangle \cdot \langle 2,3,-1\rangle = -2+6-3 = 1 > 0$ so angle is acute

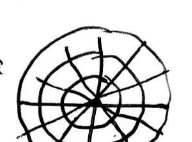
(Since $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$)

(d) If $\text{curl}\,\mathbf{F} = 0$, then $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed loop C since...

Answer 1: $\nabla\times\vec{F} = \vec{0}$ implies $\vec{F}$ is conservative, so $\oint_C \vec{F}\cdot d\vec{r} = f(\text{end}) - f(\text{start}) = 0$ since start = end

Answer 2: By Stokes Theorem, $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$ for any S with boundary C & since the integrand is $\vec{0}$, this integral gives 0.

(e) In polar coordinates, $dA = r\,dr\,d\theta$. Give a brief, informal indication of why the factor of 'r' appears in this formula.

Answer 1: The polar "grid" looks like  & since the sections between grid lines have larger area if r is larger, we need a factor to account for the changing area.

Answer 2: r is just the Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)}$ for the change of variables from x-y to r-θ coordinates, & this accounts for how areas change under the coordinate change.