Math F253 Final Exam Fall 2022

Name: Solutions

Student Id: ____________

Rules:

You have 120 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10 **or** 11	10	
Extra Credit	5	
Total	100	

1. (10 points)

Consider the vectors $\mathbf{v} = \langle 1, 2, 1 \rangle$ and $\mathbf{w} = \langle -4, 2, 3 \rangle$.

a. Compute the angle between the two vectors. An inverse trig function is allowable as part of your answer.

$$\vec{v} \cdot \vec{w} = -4 + 4 + 3 = 3$$

$$\|\vec{v}\|^2 = 1 + 4 + 1 = 6$$

$$\|\vec{w}\|^2 = 16 + 4 + 9 = 29$$

$$\boxed{\theta = \operatorname{acos}\left(\frac{3}{\sqrt{6}\sqrt{29}}\right)}$$

b. Find a vector that is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

$$\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ -4 & 2 & 3 \end{matrix}$$

$$\vec{v} \times \vec{w} = (6-2)\hat{i} - (3+4)\hat{j} + (2+8)\hat{k}$$

$$\boxed{= \langle 4, -7, 10 \rangle}$$

c. Compute the area of the parallelogram spanned by the two vectors. This is the parallelogram with vertices $\mathbf{0}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} + \mathbf{w}$.

$$\|\vec{v} \times \vec{w}\| = (16 + 49 + 100)^{1/2}$$

$$= \boxed{\sqrt{165}}$$

d. Find the equation of a plane passing through the point $(3, -2, 5)$ that is tangent to both $\mathbf{v}$ and $\mathbf{w}$.

$$\boxed{4(x-3) - 7(y+2) + 10(z-5) = 0}$$

2. (10 points)

Consider a temperature function $T(x, y)$ on a flat grassy meadow where x and y are measured in meters and T is measured in °C. Suppose you know that at the point $p = (5, 7)$ that $T(p) = 18$, and that

$$\frac{\partial T}{\partial x}(p) = -1^\circ C/\mathrm{m}$$
$$\frac{\partial T}{\partial y}(p) = 3^\circ C/\mathrm{m}$$

a. Estimate the temperature at the point $q = (4.5, 7.1)$.

$$dT = \frac{\partial T}{\partial x} d_x + \frac{\partial T}{\partial y} d_y \qquad d_x = -0.5,\ d_y = 0.1$$

$$= (-1)(-0.5) + 3 \cdot (0.1)$$

$$= 0.8$$

$$\rightarrow T + dT = 18.8\,^\circ C$$

b. My dog, Frog, loves flat grassy meadows. Suppose she is standing at the point p and is running with a velocity $\langle 2, 4\rangle$ m/s. What rate of change of temperature does she see?

$$\frac{\partial T}{\partial x} \cdot 2 + \frac{\partial T}{\partial y} \cdot 4 = -2 + 12 = 10\,^\circ C/s$$

c. Starting at p, in what direction **u** should Frog run in if she wanted to see the temperature increase most rapidly? Express your answer as a unit vector.

$$\vec{\nabla} T = \langle -1, 3 \rangle$$

$$\boxed{u = \left\langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle}$$

$$\|\vec{\nabla} T\| = \sqrt{10}$$

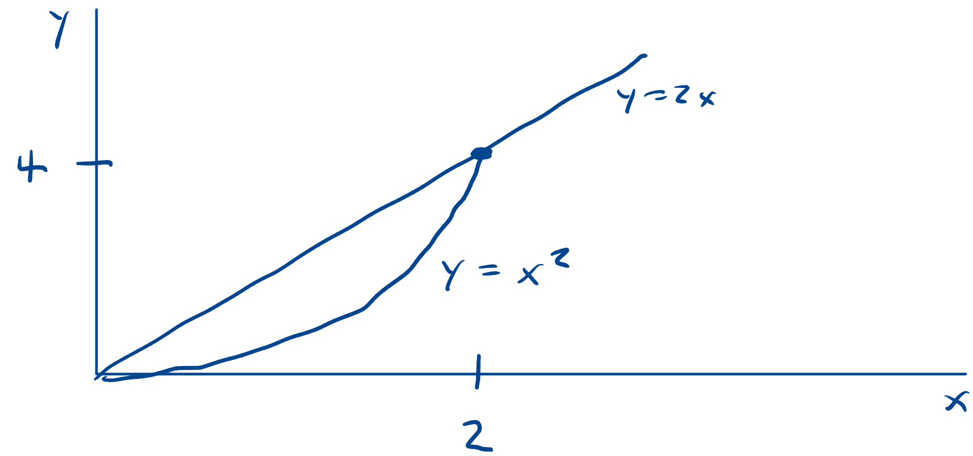
d. State a (non-zero) velocity **w** that Frog could run at, starting at p, such that Frog would initially see the temperature unchanging.

$$\boxed{\vec{w} = \langle 3, 1 \rangle}$$ since $\vec{\nabla} T \cdot \vec{w} = 0$

3. (10 points)

Consider a flat plate bounded by the lines $y = 2x$ and $y = x^2$ with x and y measured in centimeters.

a. Sketch the region, labeling any interesting points on the x- and y-axes.



b. Suppose the plate has a (planar) density $\rho(x, y) = xy$ grams per cm^2. Compute the mass of the plate.

$$\int_0^2 \int_{x^2}^{2x} xy \, dy \, dx = \int_0^2 \frac{xy^2}{2} \Big|_{x^2}^{2x} dx$$

$$= \int_0^2 \frac{x}{2} \left[4x^2 - x^4 \right] dx$$

$$= \frac{1}{2} \int_0^2 4x^3 - x^5 \, dx$$

$$= \frac{1}{2} \left(x^4 - \frac{x^6}{6} \right) \Big|_0^2$$

$$= \frac{1}{2} \left(16 - \frac{32}{3} \right) = \boxed{\frac{8}{3}}$$

4. (10 points)

Consider the vector field

$$\mathbf{X} = \left\langle z + y, x, -2z\sin(z^2) + x \right\rangle.$$

The vector field **X** is conservative. Find a potential for it.

$f_x = z + y \Rightarrow f = (z+y)x + g(y,z)$

$f_y = x + g_y$

$f_y = x \Rightarrow g_y = 7 + h(z)$

$\Rightarrow f = (z+y)x + 7 + h(z)$

$\Rightarrow f_z = x + h'(z)$

$f_z = -2z\sin(z^2) + x \Rightarrow h'(z) = -2z\sin(z^2)$

$$\boxed{f = (z+y)x + 7 + \cos(z^2)}$$

5. (10 points)

A particle has velocity

$$\mathbf{v}(t) = \langle \sin(4t), \cos(4t), e^{-t} \rangle$$

for $t \geq 0$. At time $t = 0$ the particle is at the origin. Compute the position $\mathbf{r}(t)$ of the particle for $t \geq 0$.

$\vec{r}(t) = \int v(t) = \langle -\frac{1}{4}\cos(4t), \frac{1}{4}\sin(4t), -e^{-t} \rangle + \vec{c}$

$\vec{r}(0) = \langle -\frac{1}{4}, 0, -1 \rangle + \vec{c}$

$\vec{r}(0) = \langle 0, 0, 0 \rangle \Rightarrow \vec{c} = \langle \frac{1}{4}, 0, 1 \rangle$

$\vec{r}(t) = \langle \frac{1}{4}(1 - \cos(4t)), \frac{1}{4}\sin(4t), 1 - e^{-t} \rangle$

6. (10 points)

Consider the function

$$f(x, y) = x^3 + 3xy + y^3.$$

a. Find all critical points of this function

$f_x = 3x^2 + 3y$

$f_y = 3x + 3y^2$

$f_x = 0 \Rightarrow x^2 = -y$

$f_y = 0 \Rightarrow 3x + 3x^4 = 0$

$\Rightarrow x(1 + x^3) = 0$

$\Rightarrow x = 0, -1$

$\Downarrow$ $y = 0$ $\Downarrow$ $y = -1$

$\boxed{(0,0) \quad (-1,-1)}$

b. Determine, with justification, whether each critical point is a local minimum, local maximum or saddle.

$f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = 3$

$H = f_{xx} f_{yy} - f_{xy}^2 = 36xy - 9$

$(0,0)$: $H = -9 \Rightarrow$ saddle

$(-1,-1)$: $H = 36 - 9 = 27 > 0$, $f_{xx} < 0$ $\Rightarrow$ local max

7. (10 points)

Suppose we have an electric field

$$\mathbf{E} = \langle x, y, 1 - z \rangle$$

in newtons per coulomb and that a charged particle moves along a path

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

for $0 \le t \le 4\pi$. Note that position coordinates are measured in meters and that time is measured in seconds.

a. Compute $\mathbf{r}'(0)$ and interpret this quantity physically.

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$\vec{r}\,'(0) = \langle 0, 1, 1 \rangle \ m/s$$

velocity of particle at $t = 0$

b. The particle has charge of 2 coulombs and hence the force acting on the particle is $\mathbf{F} = 2\langle x, y, 1 - z \rangle$ newtons. Compute the work done by this force between $t = 0$ and $t = 4\pi$.

$$\int_0^{4\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{4\pi} 2\langle \cos(t), \sin(t), 1 - t \rangle \cdot \langle -\sin(t), \cos(t), 1 \rangle\,dt$$

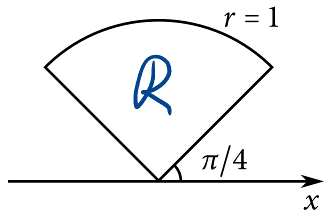
$$= \int_0^{4\pi} 2\left(-\sin(t)\cos(t) + \sin(t)\cos(t) + 1 - t\right)dt$$

$$= 2\int_0^{4\pi} 1 - t\,dt = 2\left(t - \frac{t^2}{2}\right)\Big|_0^{4\pi}$$

$$= \boxed{8\pi - 16\pi^2}$$

8. (10 points)

Consider the wedge below where $\pi/4 \le \theta \le 3\pi/4$ and $0 \le r \le 1$.



Use Green's theorem to compute

$$\int_C -y^2\,dx + x\,dy.$$

$F = \langle P, Q \rangle = \langle -y^2, x \rangle$

$$\iint_R Q_x - P_y \, dA = \iint_R (1+2y)\,dx\,dy$$

$$= \int_0^1 \int_{\pi/4}^{3\pi/4} (1 + 2r\sin\theta)\, d\theta \, r\,dr$$

$$= \int_{\pi/4}^{3\pi/4} \int_0^1 r + 2r^2 \sin\theta \, dr \, d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \frac{r^2}{2} + 2\frac{r^3}{3}\sin\theta \Big|_0^1 \, d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \frac{1}{2} + \frac{2}{3}\sin\theta \, d\theta$$

$$= \frac{\theta}{2} - \frac{2}{3}\cos\theta \Big|_{\pi/4}^{3\pi/4} = \boxed{\frac{\pi}{2} + \frac{2\sqrt{2}}{3}}$$

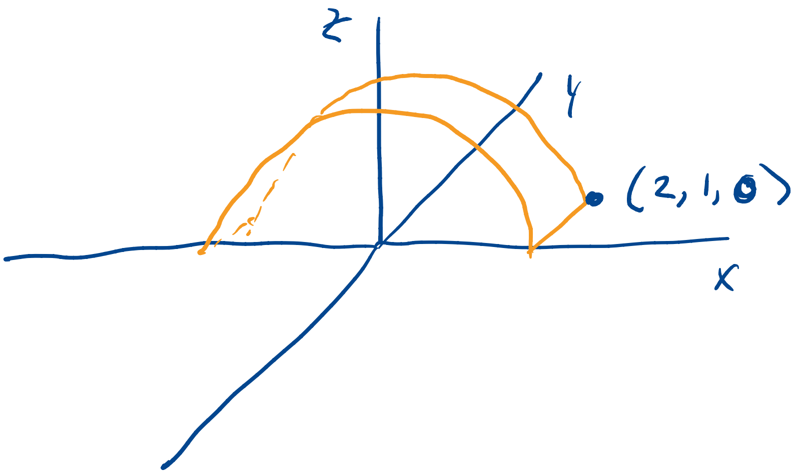
9. (10 points)

Consider a surface $\mathcal{S}$ parameterized by

$$\mathbf{r}(u, v) = \left\langle u, v, 4 - u^2 \right\rangle$$

where $-2 \leq u \leq 2$ and where $0 \leq v \leq 1$.

a. Sketch this surface.



b. Find the **unit** normal (pointing generally in the positive z direction) of the surface at the point $(1, 1, 3)$.

$$\vec{r}_u = \langle 1, 0, -2u \rangle$$

$$\vec{r}_v = \langle 0, 1, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 2u, 0, 1 \rangle$$

$$= \langle 2, 0, 1 \rangle \quad @ (1, 1, 3)$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{5}$$

$$\boxed{\vec{n} = \left\langle \frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \right\rangle}$$

Continued....

Problem 9 continued....

Recall that we are working with a surface $\mathcal{S}$ parameterized by

$$\mathbf{r}(u,v) = \left\langle u, v, 4 - u^2 \right\rangle$$

where $-2 \leq u \leq 2$ and where $0 \leq v \leq 1$.

c. Consider a vector field $\mathbf{Z} = \langle xz, 1, 0 \rangle$. Set up, but **DO NOT EVALUATE**, the flux integral

$$\iint_{\mathcal{S}} \mathbf{Z} \cdot \mathbf{n} \, dS$$

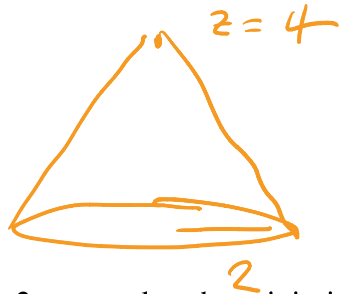
where $\mathbf{n}$ is the normal pointing generally in the positive z direction. For full credit, your answer should be an iterated integral with explicit endpoints of integration and with an integrand that is an explicit function of u and v.

$$\int_{-2}^{2} \int_{0}^{1} \langle u(4-u^2), 1, 0 \rangle \cdot \langle 2u, 0, 1 \rangle \, du \, dv$$

$$\boxed{\int_{-2}^{2} \int_{0}^{1} 2u^2(4-u^2) \, du \, dv}$$

d. In fact, in this problem position (x, y and z) was measured in meters and the vector field $\mathbf{Z}$ was the mass flux vector field $\rho\mathbf{v}$ of a fluid, where the density ρ was measured in kg per m^3 and where the velocity $\mathbf{v}$ of the fluid was measured in meters per second. In a sentence or two (that your parents would understand), explain the physical meaning, including units, of the quantity you computed in part **(c)**.

The rate in kg/s fluid is passing through $\mathcal{S}$ in the direction of $\vec{n}$.



10. (10 points)

Complete **EITHER** this problem or problem **(11)**.

Consider a solid cone $\mathcal{E}$ with its base a circle of radius 2 centered at the origin in the xy plane and with total height $h = 4$, so the vertex of the cone lies at the point $(0, 0, 4)$.

a. Compute the height z of the cone as a function of the distance $r = \sqrt{x^2 + y^2}$. Your answer should be a function $z(r)$ with, for example, $z(0) = 4$ since when $r = 0$ the height of the cone is 4.

$$z(r) = 4 - 2r$$

b. Consider the vector field $\mathbf{Z} =< 2x, 2y, -z^3/3 >$. Compute the divergence $\nabla \cdot \mathbf{Z}$.

$$\operatorname{div} Z = 2 + 2 - z^2$$
$$= 4 - z^2$$

c. The divergence theorem states that the flux of $\mathbf{Z}$ to the exterior of the cone can be computed by a volume integral over the interior of $\mathcal{E}$. Set up, but **DO NOT EVALUATE**, this integral. For full credit, you should use cylindrical coordinates to write down an iterated integral, and all endpoints of integration need to be fully specified.

$$\boxed{\iiint_{\mathcal{E}} \operatorname{div} Z \, dV = \int_0^{2\pi} \int_0^{1} \int_0^{4-2r} 4 - z^2 \, dz \; r \, dr \, d\theta}$$

11. (10 points)

Complete **EITHER** this problem or problem **(10)**.

Consider the surface S

$$x^2 + y^2 + z^2 = 6$$

with $z \geq 0$ and oriented so that its unit normal points away from the origin. Let $\mathbf{Z} = \left\langle \sqrt{y}\sin(z), x, xy \right\rangle$. Recall that Stokes' Theorem says that

$$\iint_S \nabla \times \mathbf{Z} \cdot \mathbf{n}\, dS = \int_C \mathbf{Z} \cdot d\mathbf{r}$$

so long as $\mathbf{n}$ and C are consistently oriented.

a. 4 points Compute the curl $\nabla \times \mathbf{Z}$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \sqrt{y}\sin(z) & x & xy \end{vmatrix}$$

$$\nabla \times \vec{Z} = (x-0)\hat{i} - (\sqrt{y}\cos(z) - 1)\hat{j} + \left(1 - \frac{1}{2\sqrt{y}}\sin z\right)\hat{k}$$

$$\boxed{= \left\langle x, 1 - \sqrt{y}\cos(z), 1 - \frac{1}{2\sqrt{y}}\sin(z) \right\rangle}$$

b. 6 points Use Stokes' Theorem to determine the flux $\iint_S \nabla \times \mathbf{Z} \cdot \mathbf{n}\, dS$ by computing a **line integral** instead. Hint: because $\sin(0) = 0$, your integrand should be pretty simple!

$$\vec{r}(t) = \langle \sqrt{6}\cos(t), \sqrt{6}\sin(t), 0 \rangle$$

$$Z(\vec{r}(t)) = \langle 0, \sqrt{6}\cos t, 6\cos(t)\sin t \rangle$$

$$\vec{r}\,'(t) = \langle -\sqrt{6}\sin(t), \sqrt{6}\cos(t), 0 \rangle$$

$$Z \cdot \vec{r}\,' = 6\cos^2(t)$$

$$\int_0^{2\pi} Z \cdot \vec{r}\,' dt = \int_0^{2\pi} 6\cos^2(t) = 6 \cdot \frac{1}{2} \cdot 2\pi = \boxed{6\pi}$$

Extra credit (5 points):

The curve $\mathbf{r}(t)$ in problem **(5)** is a fun one. Sketch the curve and describe, in words, its long term behaviour at $t \to \infty$. Be as precise as you can. For example, if a line is involved, its direction and a point on it should be specified. If a circle involved, its full geometry, including its orientation should be specified. Etc.

$$\vec{r}(t) = \Big\langle \underbrace{\tfrac{1}{4}(1-\cos(4t)),\ \tfrac{1}{4}\sin(4t)}_{\text{circle of radius } 1/4 \text{ centered at } (1,0)},\ 1-e^{-t} \Big\rangle$$

height rises and asymptotes to 1