Last class:

critical point: $\vec{\nabla} f = 0$ or DNE.

At a local min/max in interior of domain,
we have a crit point.

So if looking for max/min, in interior
need only look at critical points.

for $f(x,y)$ (2-d) we have a 2nd deriv test

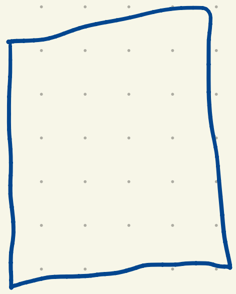
$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad D = |H| = f_{xx} f_{xy} - (f_{xy})^2$$

If $D > 0 \Rightarrow$ local min/max

$D < 0 \Rightarrow$ saddle

$D = 0 \Rightarrow$ inconclusive

$f_{xx} > 0 \Rightarrow$ local min (f_yy also)
$f_{xx} < 0 \Rightarrow$ local max



closed bounded domain.

(includes boundary)

fits in a box.

A continuous function on such a domain will attain a max/min.

This happens either at

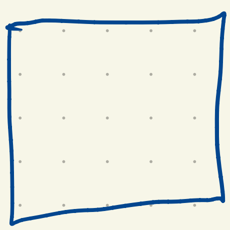
1) an interior critical point

2) on the boundary.

Trouble with boundaries:

min/max can happen on boundary

If f is cts and domain is bounded and closed it attains a max/min:



$$f(x,y) = x^2 - 2xy + 2y$$

$$0 \le x \le 3$$
$$0 \le y \le 2$$

$$\frac{\partial f}{\partial x} = 2x - 2y \Rightarrow x = y$$

$$\frac{\partial f}{\partial y} = -2x + 2 \Rightarrow x = 1$$

$\Rightarrow 1,1$

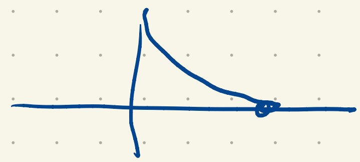
$$f(1,1) = 1 - 2 + 2 = 1$$

On $x = 0$ is $2y$ $\quad f(0,2) = 4$

On $x = 3$ is $9 - 6y + 2y = 9 - 4y$ $\quad y = 0,\ 9.$

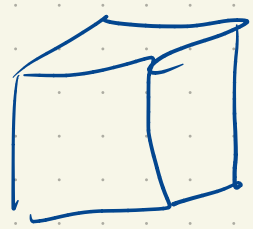
On $y = 0$ is x^2 $0 \le x \le 3$ is 9

On $y = 2$ $x^2 - 4x + 4$ $0 \le x \le 2$

$(x-2)^2$  max at 4,

Min is 0.

Graphing matlab,

$V = xyz$ 

$x+y+z \le 96$ (shippers reg)

Task: maximize volume given constraint.

Given x, y $z \le 96 - x - y$.

Make as big as possible: $z = 96 - x - y$.

So: $V = xy(96-x-y)$.

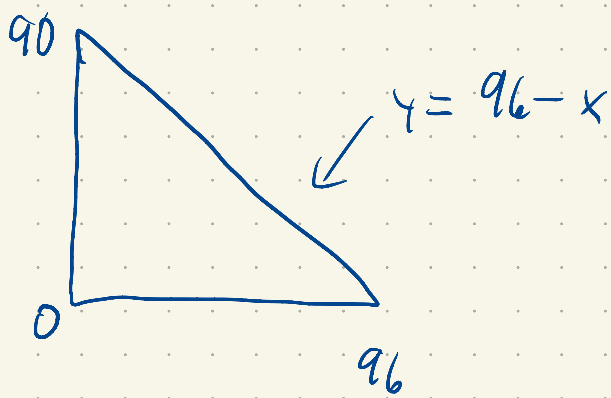
With constraints: $x \geq 0$

$y \geq 0$

$z \geq 0 \Rightarrow 96-x-y \geq 0$

$\Rightarrow 96 \geq x+y$

$y \leq 96-x$



$V=0$ on boundary.

$\nabla V = 0$

$$\frac{\partial V}{\partial x} = y(96-x-y) - xy = y[96-2x-y]$$

$$\frac{\partial V}{\partial y} = x(96-x-y) - xy$$

$$= x[96-x-2y]$$

$\frac{\partial V}{\partial x} = 0$ at $y = 0$ or $96 - 2x - y = 0$

$\frac{\partial V}{\partial y} = 0$ at $x = 0$ or $96 - x - 2y = 0$

Subtract: $-x + y = 0 \Rightarrow y = x$

$$96 - 3x = 0$$

$$x = 32$$

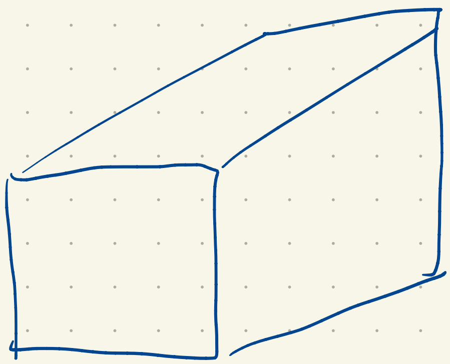
$$y = 32$$

$$z = 96 - x - y = 32.$$

It's a cube!

Graph in matlab.

Section Lagrange Multipliers



$V = xyz$

girth + length ≤ 108

$2x + 2y + z \leq 108$

clearly an increase, so

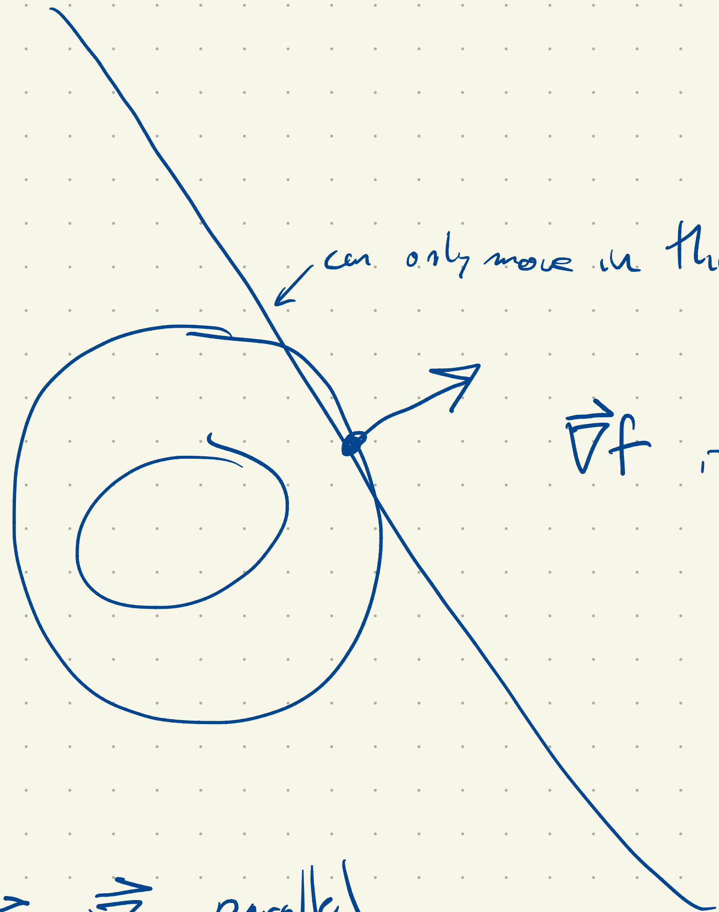
Maximize $V = xyz$ subject to

$$\underbrace{2x + 2y + z = 108}_{\text{constraint}},$$

We'll come back to this.

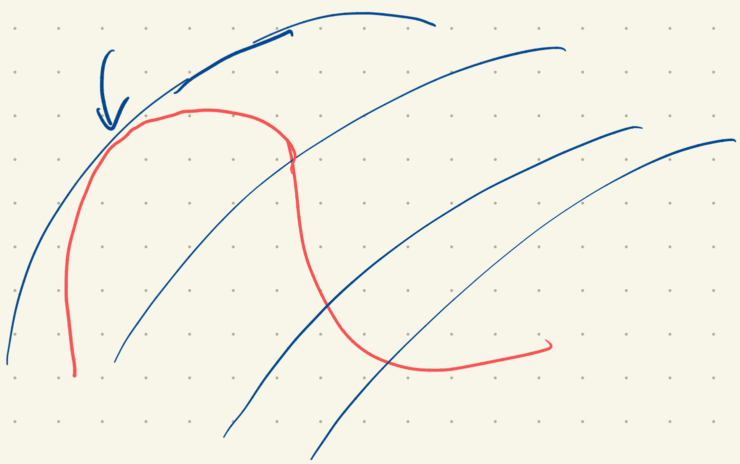
Let us instead minimize

$f(x,y) = x^2 + y^2$ subject to $\underbrace{x + y}_{g(x,y)} = 9$



$\vec{\nabla} f$ is parallel to $\vec{\nabla} g$

$\vec{\nabla} f$, $\vec{\nabla} g$ parallel



At a maximizer

$$\vec{\nabla} f(x_0,y_0) = \lambda \vec{\nabla} g(x_0,y_0)$$

3 eq's for 3 unknowns.

$$\left[\begin{array}{l} g(x_0,y_0) = 9 \quad (c \text{ in general}) \\ f_x(x_0,y_0) = \lambda g_x(x_0,y_0) \\ f_y(x_0,y_0) = \lambda g_y(x_0,y_0) \end{array}\right.$$

$$\vec{\nabla} f = \langle 2x, 2y \rangle$$

$$\vec{\nabla} g = \langle 1, 1 \rangle$$

$$x + y = 9$$

$$\left.\begin{array}{l} 2x = \lambda \\ 2y = \lambda \end{array}\right] \quad x = y = \frac{\lambda}{2}$$

$$\lambda + \lambda = 9 \Rightarrow \boxed{x = 9/2 \ , \ y = 9/2}$$

($\lambda = 9$ is not essential)

$$f(9/2, 9/2) = \frac{81}{4} \cdot 2 = \frac{81}{2}$$

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e.g. Find extreme values of

$x^2 + 4y^3$ on the ellipse $x^2 + 2y^2 = 1$

$$\nabla f = \langle 2x, 12y^2 \rangle$$

$$\nabla g = \langle 2x, 4y \rangle$$

$2x = \lambda 2x$

$12y^2 = \lambda \cdot 4y$

$x^2 + 2y^2 = 1$

$\lambda = 1$ or $x = 0$

$3y^2 = y$

$y = \frac{1}{3}, 0$

$2y^2 = 1$

$\Rightarrow y = \pm \frac{1}{\sqrt{2}}$

$\lambda = 0$ unimportant

$x^2 + \frac{2}{9} = 1$

$x = \pm \frac{\sqrt{7}}{3}$

$\left(\pm \frac{\sqrt{7}}{3}, \frac{1}{3}\right)$

$y = 0$

$x = \pm 1$

$(\pm 1, 0)$

$\left(0, \pm \frac{1}{\sqrt{2}}\right)$

contour: x^2+4y^3

$x^2+2y^2=1$

evaluate $f(1,0)=f(-1,0)=1$

$f\left(\frac{\sqrt{7}}{3},\frac{1}{3}\right)=f\left(-\frac{\sqrt{7}}{3},\frac{1}{3}\right)=\frac{7}{9}+\frac{4}{27}=\frac{25}{27}$

$f\left(0,\frac{1}{\sqrt{2}}\right)=\sqrt{2}$ ← max

$f\left(0,-\frac{1}{\sqrt{2}}\right)=-\sqrt{2}$ ← min

For functions of 3 variables

- $F(x,y,z)$ $\quad g(x,y,z)=c$

maximize

$$\vec{\nabla}F(x_0,y_0,z_0)=\lambda\vec{\nabla}g(x_0,y_0,z_0) \rightarrow \partial_x F=\lambda\partial_x g \text{ etc.}$$

$$g(x,y,z)=c$$

4 eq's for 4 unknowns $(x_0,y_0,z_0),\lambda$.

$V = xyz \qquad 2x + 2y + z = 108$

$V_x = yz \qquad g_x = 2$

$V_y = xz \qquad g_y = 2$

$V_z = xy \qquad g_z = 1$

$yz = 2\lambda$

$xz = 2\lambda \qquad 2x + 2y + z = 108$

$xy = \lambda$

$yz = 2xy \qquad z = 2x \quad (y \neq 0)$

$xz = 2xy \qquad z = 2y \quad (x \neq 0)$

$3z = 108$

$z = 36$

$x = 18$

$y = 18$