3 facts from calc I

a) If $f(x)$ attains a local max/min on $[a,b]$ but not at an endpoint, $f'(x)=0$ there

b) If $f'(x)=0$ and $f''(x)>0 \Rightarrow$ local min

$f''(x)<0 \Rightarrow$ local max

New versions

$f(x_0,y_0) \geq f(x,y)$

for all (x,y) close to (x_0,y_0).

a) $f(x,y)$

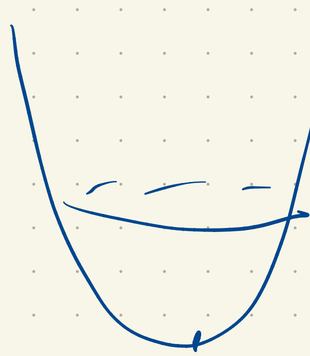
If we attain a local max/min at (x_0,y_0) in interior and derivatives exist there

$$\frac{\partial f}{\partial x}(x_0,y_0)=0 \qquad \frac{\partial f}{\partial y}(x_0,y_0)=0$$

$\left(\vec{\nabla} f(x_0, y_0) = \vec{0} \right)$

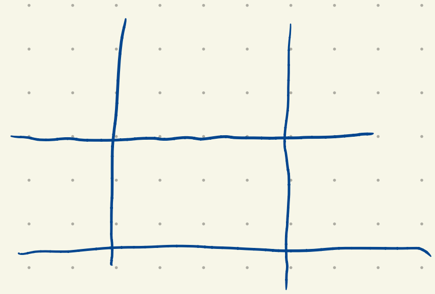
$f(x,y,z)$ $\left(\frac{\partial f}{\partial z} = 0 \text{ also} \right)$

We saw this



Critical point: $f_x = 0$ and $f_y = 0$

(or DNE)



E.g. $f(x,y) = xy(x-2)(y+3)$

Find crit pts

$f_x = y(x-2)(y+3) + xy(y+3) = y(y+3)\left[2x-2\right]$

$$f_y = x(x-2)(y+3) + xy(x-2) = x(x-2)\left[2y+3\right]$$

$f_x = 0$ $y=0$, $y=-3$, $x=1$	$f_y = 0$ $x=0$, $x=2$, $y=-\frac{3}{2}$

$y=0$ $x=0,\ 2$ $(1,0)$ $(0,0)$

$y=-3$ $x=0,\ x=2$ $(0,3)$ $(2,3)$

$x=1$ $y=-3/2$ $(1,-3/2)$

So: there are 5 possible locations for min/max.

How can we determine what kind?

Models: x^2+y^2 $-x^2-y^2$ x^2-y^2

min   max   saddle

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Hessian matrix

symmetric

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad |\cdot| = 4 > 0$$

diagonal elements are > 0

$$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad |\cdot| = 4 > 0$$

diagonal elements are < 0

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \qquad |\cdot| = -4 < 0$$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$D > 0$, diag $> 0 \Rightarrow$ local min

, diag $< 0 \Rightarrow$ local max

$D < 0 \Rightarrow$ saddle point

$D = 0$ inconclusive (like $f''(x) = 0$).

$$f_x = y(x-2)(y+3) + xy(y+3) = y(y+3)[2x-2]$$

$$f_y = x(x-2)(y+3) + xy(x-2) = x(x-2)[2y+3]$$

$$f_{xx} = 2(y)(y+3) \qquad f_{xy} = 2(x-1)[2y+3]$$

$$f_{yy} = 2(x)(x-2)$$

$$D = 4xy(x-2)(y+3) - 4(x-1)^2(2y+3)^2$$

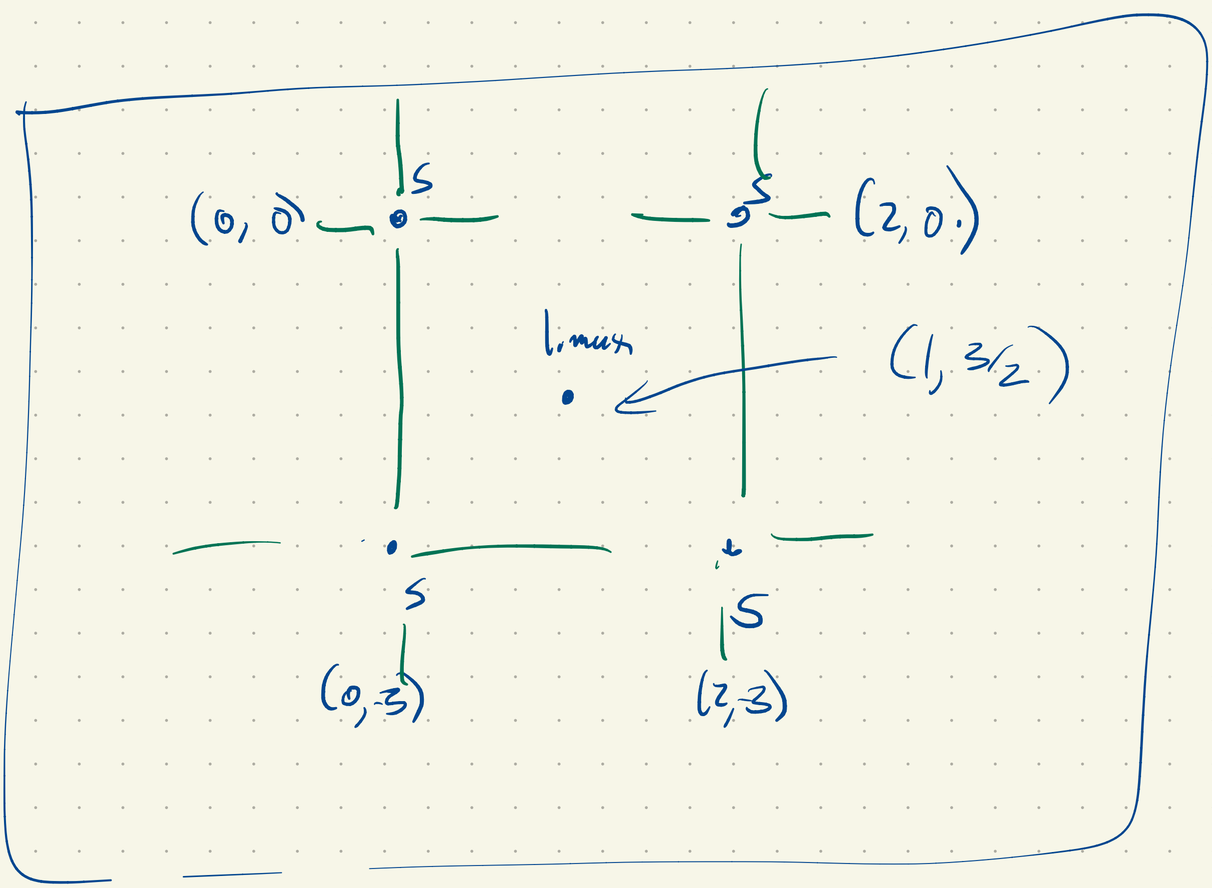
$D(1, -\frac{3}{2}) = 1 \cdot (-3)(1-2) \cdot (-3+6) - 4 \cdot 0$

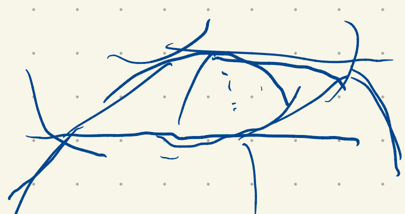
$= 9 > 0 \qquad f_{yy} = -2 < 0$

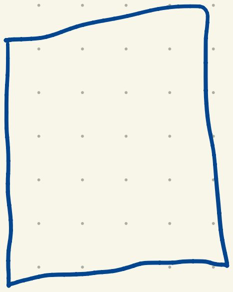
$\Rightarrow$ local max

$D(0,0) = -4 \cdot 1 \cdot 9 = -36$

$\Rightarrow$ saddle







closed bounded domain.

(includes boundary)

fits in a box.

A continuous function on such a domain will attain a max/min.

This happens either at

1) an interior critical point

2) on the boundary.

E.g. Maximize $V = xyz$ subject to $x, y, z \geq 0$

$$x + y + z \leq 96$$

$$z \leq 96 - x - y$$

(skipping negs!)

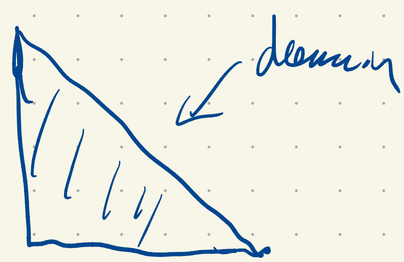
$$z = 96 - x - y$$

$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 96$$

$$V = xy(96 - x - y)$$

$$V_x = y(96 - x - y) - xy$$

$$= y(96 - y) - 2xy = y[96 - y - 2x]$$



$$V_y = x(96 - x) - 2xy = x[96 - x - 2y]$$

$V_x = 0$: $y = 0$ or $96 - y - 2x = 0$

$V_y = 0$: $x = 0$ or $96 - x - 2y = 0$

$(0,0)$, $(0,96)$ $(96,0)$

$$-y + x - 2x + 2y = 0$$

$$-x + y = 0 \qquad x = y$$

$$96 - x - 2x = 96 - 3x$$

$$x = 32$$

$$y = 32$$

$$V_{xx} = -2y \qquad V_{yy} = -2x \qquad V_{xy} = 96 - 2y - 2x$$

$$D = V_{xx}V_{yy} - (V_{xy})^2 = 4xy - (96 - 2x - 2y)^2$$

$$D(32,32) = 3072 > 0$$

$V_{xx} < 0 \Rightarrow$ local max

whew!

$$z = 96 - 32 - 32 = 32 \ (!)$$

Cube: 32^3

Last class:

critical point: $\vec{\nabla} f = 0$ or DNE.

At a local min/max in interior of domain,
we have a crit point.

So if looking for max/min, in interior
need only look at critical points.

for $f(x,y)$ (2-d) we have a 2nd deriv test

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad D = |H| = f_{xx}f_{yy} - (f_{xy})^2$$

If $D > 0 \Rightarrow$ local min/max

$D < 0 \Rightarrow$ saddle

$D = 0 \Rightarrow$ inconclusive

$f_{xx} > 0 \Rightarrow$ local min (f_{yy} also)
$f_{xx} < 0 \Rightarrow$ local max